TECHNICAL REPORT

MATHEMATICAL MODELS FOR NAVIGATION SYSTEMS

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ABSTRACT

The principal objective of this study was an evaluation of the formulas basic to the geodetic inverse solution for distance computations used by the U. S. Naval Oceanographic Office in loran-type charting. The adequacy of the formulas for past requirements was verified but, in anticipation of future requirements, they were modified to give geodesic distances and azimuths between any two points on the reference ellipsoid to uncertainties of less than a meter and a second respectively.

During the study, associated geometrical configurations were developed or studied: latitudes associated with the auxiliary sphere-spheroid configuration; a spherical rectangular coordinate system on the auxiliary sphere with hyperbolic loci referenced to it; and geometrical quantities associated with arc distance, such as chord length, dip of the chord, maximum separation of chord and arc, and the geographical position of the point of maximum separation. The formulas with their derivations are presented.

FOREWORD

Increased knowledge of tropospheric and ionospheric effects on electromagnetic propagation, gained through artificial satellite experiments and related studies, foreshadows an increased accuracy requirement in the geodetic parameters involved in computing charts and tables for electromagnetic navigational and positioning systems. This report examines some of the mathematical models involved in these computations, verifying their adequacy for past requirements and introducing modifications to improve range and accuracy capabilities.

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PREFACE

Early in World War II, the U. S. Navy Hydrographic Office began publishing charts and tables for the new loran system of long-range radio navigation. Loran and similar systems make use of radio waves to measure earth distances or distance-differences (hyperbolas) for positioning ships or aircraft at long ranges from the shore transmitter stations. The computation of these navigational lines of positioning is a problem in geodesy. Because of the irregularities of the shape of the actual earth over which radio waves travel, geodesists are forced to use mathematical models that approximate the shape of the earth when computing navigational lines of position.

Various models and co-ordinate systems have been used in making loran-type computations, which were originally done by desk calculators within limits of accuracy compatible with the early navigation systems. Now, however, improved system accuracies and better information of the figure of the earth have made necessary a re-examination of the mathematical formulas to ensure their adequacy at very long ranges.

The inverse distance formula used in loran computations is actually the so-called Andoyer-Lambert approximation and is the expansion of the geodesic arc length between two points on the reference ellipsoid to first order in the flattening. There are two simple and very similar forms of the approximation, one in terms of geodetic latitude and the other in terms of parametric latitude. The U. S. Naval Oceanographic Office uses the latter which requires a conversion from geodetic latitude. While the parametric form gives slightly more accurate distance computations, the objective of this study was to determine whether the latitude conversions are justified and to investigate the second-order terms in the expansions and their contribution to the accuracy of the computations.

It was the conclusion of the study that the parametric formulas which have been used are in fact adequate to meet present operational requirements but that the conversion to parametric latitude is not necessary. In anticipation of future requirements, the geodetic formulas were extended to give geodesic distances and azimuths between any two points on the reference ellipsoid to uncertanties of less than a meter and a second respectively, out to ranges of 6000 miles.

During the investigation, formulas were developed for the particular quantities involved and were transformed in terms of particular computational parameters. Some associated useful geometrical quantities were included relative to distance computations: chord distance, the dip of the chord, the maximum separation distance between chord and arc (surface), and the geographic position of the point where maximum separation occurs. Some of these relationships can be found

in accessible sources, but many are not readily available. Some are new, such as the expansion of the geodesic to second order in the flattening in both geodetic and parametric latitudes, and conversion formulas between the two forms.

Since the entire treatment is mathematical, an overall summary of the investigation is first presented to allow a quick assay of the topics and accomplishments. This summary is followed by a condensation of the formulas developed or included. The details of the actual development follow in three sections with computational examples in the Appendices.

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MATHEMATICAL MODELS
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OVERALL SUMMARY OF INVESTIGATIONS

Latitude

A loran station positioned on the auxiliary sphere of the ellipsoid of reference has as its geodetic latitude the angle at the equator made by that normal to the meridian which passes through the station of the sphere. Its longitude will remain the same. See Figure 1, page 13. Now this is analogous to the geodetic latitude of a subsatellite point, if the trajectory were confined wholly to the surface of the auxiliary sphere. It is clearly not one of the three commonly associated latitudes as shown in equation (1), page 12. Actually the relationship between geocentric latitude on the sphere and geodetic latitude on the ellipsoid is given by equation (2), page 12. From these one may find the maximum value of the difference, $\Delta\phi$, and the value of ϕ, the geodetic latitude, at which this maximum difference occurs, equations (3) – (6), page 14. The expansions of $\Delta\phi$ in series in terms of ϕ were obtained and are given in equations (7) – (20), pages 15, 16.

For computation of ϕ as a function of θ, the geocentric latitude, it was necessary to employ the Lagrange expansion formula and the resulting expansion and formulas are given in equations (21) – (33), pages 16 to 18. Development of the series expansions for the height, h, of the auxiliary sphere above the ellipsoid is given in equations (43)– (48). See Figure 1, page 13 and pages 19, 20. A summary of latitude formulas and a bibliography of pertinent references are included, pages 21 – 22.

The great circle track as determined by the geographical coordinates of two given points on the auxiliary sphere. Parallels at a given distance from a great circle track.

As shown in figure 2, page 24, the treatment is somewhat different than the usual one in that one works from the vertex of the great circle or the point where the great circle is orthogonal to a meridian. This simplifies computations and provides well balanced triangles from which to compute. It also facilitates the computations for parallels at a given distance from a fixed great circle track as shown in Figures 3 and 4, pages 26 and 27. See also equations (1) – (13), pages 23–27.

A spherical rectangular coordinate system with a great circle base line as an axis.

Figure 5, page 29, shows, pictorially, this coordinate system developed on the great circle base line referenced to the vertex of the great circle base line. The conversion equations are developed in equations (14) to (26), pages 28 to 30.

Derivation of the equations of spherical hyperbolas and their plane equivalents.

Having established a spherical rectangular coordinate system we are in a position to derive the equations of spherical hyperbolas referenced to the system. This is done in both spherical rectangular coordinates and spherical polar form, equations (27) to (50), pages 31 to 34. See also figures 5, 6, and 7, pages 29, 32, 34.

The plane hyperbola equivalents are developed in equations (51) to (59), pages 35 and 36 and a comparison table of the spherical and plane equivalents is given as equation (60), page 37. See also Figures (8) and (9), pages 35 and 36.

An example of computations using the plane hyperbola approximation is given as Appendix 1, pages 99 to 110.

Distance computations and conversions; Azimuths; Associated geometrical quantities.

The classical "inverse" problem of geodesy was considered here since it is inherent in the electronic navigational systems problem. In the "inverse" problem, the latitudes and longitudes of each of two points are given from which the distance between the points and the azimuths at the two given points are to be determined.

The geodesic on the reference ellipsoid, other than meridians and circular equator, is a space curve, and its vertex (the latitude where it is orthogonal to a meridian) is not easily expressible in terms of the geographical coordinates (latitude and longitude) of two points on it. The actual length involves the evaluation of an elliptic integral, whose modulus depends on the latitude of the vertex of the geodesic. Iterative solutions have been devised as Helmert's, based on the earlier work of Bessel.

Approximations based on plane curves which are near the geodesic in length as the normal sections and the great elliptic arc have been devised. An investigation of these was made, including some extensions for instance in the series development for the great elliptic arc approximation. See pages 48 to 51 and Figure 15, page 50. Also their use and expression in terms of common computational parameters with some associated geometrical quantities useful in operational applications as the angle of depression of the chord below the horizon, the maximum separation between the chord and the surface, and the geographic coordinates of the point on the surface where maximum separation occurs.

An investigation of the expansion of the geodesic length in powers of the flattening was made which to first order in the flattening are the well-known, so-called Andoyer-Lambert

approximation formulas, one in terms of parametric latitude, the other in terms of geodetic latitude. Since this Office uses the Andoyer-Lambert form in terms of parametric latitude, in which geographic latitudes must first be converted to parametric, an investigation was made to see if use of the parametric form to first order in the flattening was justified or necessary in terms of operational requirements. This was done in connection with a review of an extensive study by USAF (ACIC) of geodetic lines up to 6000 miles in length where the Andoyer-Lambert approximation was recommended for such tasks as LORAN computing, since the errors in the very near geodetic distances obtained are fairly constant on lines 50 to 6000 miles in length and in all azimuths. The comparisons are given in tables 1 – 3, pages 65 to 67.

Since some of the excursions in the first order form were of the order of 10 meters, the problem of obtaining the expansion of the geodesic to second order terms in the flattening was examined. By introducing two parameters X and Y, in terms of the latitude of the vertex of the great elliptic arc, it was found that the great elliptic arc approximation produced the so-called Andoyer-Lambert first order approximations. (See pages 68 – 69.) Similarly they could be produced by modification of the differential equation to the geodesic (See pages 69 to 74).

In review of an 1895 paper by the British Mathematician, A. R. Forsyth, by identifying his fundamental approximation parameter as the vertex of the great elliptic arc, it was found that he actually had both so-called Andoyer-Lambert first order expansions in the flattening, but it had apparently not been recognized. Furthermore, he had an expansion to second order terms in the flattening and in terms of geodetic latitude but it had two errors in the second order term. After these had been detected and corrected, computations based on the resulting equations give distances within a meter on all lines computed from 50 to 6000 miles. See pages 75 to 81.

Forsyth did not have the expansion to the geodesic in terms of parametric latitude to second order terms in the flattening, so his results were extended to second order terms. See pages 79 to 90. Then transformation equations were developed to convert one form to the other as far as second order terms in the flattening, pages 90 to 92, and finally the difference formulas for the principal parameters, pages 92 to 93. As a result of this study, distance and azimuth formulas are available in terms of easily computed parameters, in terms of either parametric or geodetic latitude which will give distances over all lines within a meter and azimuths within a second which is adequate for any operational requirement. A more detailed summary of the investigations of this section with a bibliography of references is given on pages 93 to 97.

NEW LATITUDE FORMULAS

If θ is the geocentric latitude of a point $P(a\cos\theta, a\sin\theta)$ on the auxiliary sphere, then the corresponding geodetic latitude ϕ of P at an altitude h above the ellipsoid of reference as shown in Figure 1, is given by

$$\sin\Delta\phi = \sin(\phi-\theta) = (e^2/2a)\, N\sin 2\phi = (e^2\sin\phi\cos\phi)/(1-e^2\sin^2\phi)^{1/2}$$

$$= c_1\sin 2\phi - c_2\sin 4\phi + c_3\sin 6\phi - c_4\sin 8\phi,$$

$$c_1 = (e^2/2) + (e^4/8) + (15e^6/256) + (35e^8/1024)$$

$$c_2 = (e^4/16) + (3e^6/64) + (35e^8/1024),$$

$$c_3 = (3e^6/256) + (15e^8/1024),$$

$$c_4 = 5e^8/2048$$

With the same coefficients,

$$\phi - \theta = \Delta\phi\ (\text{radians}) = (c_1 + c_1^3/8)\sin 2\phi - (c_2 + c_1^2c_2/4)\sin 4\phi + (c_3 - c_1^3/24)\sin 6\phi$$

$$\Delta\phi\ (\text{seconds}) = (206{,}264.8062)\cdot\Delta\phi\ (\text{radians}).$$

To express $\Delta\phi$ in terms of θ, we have

$$\tan\phi = \tan\theta + (e^2/a\cos\theta)\, N\sin\phi$$

$$= \tan\theta + (e^2/\cos\theta)\sin\phi/(1 - e^2\sin^2\phi)^{1/2},$$

which, when expanded by the Lagrange expansion formula gives

$$\Delta\phi = \phi - \theta = c_1\sin 2\theta + c_2\sin 4\theta + c_3\sin 6\theta + c_4\sin 8\theta$$

$$c_1 = (e^2/2) + (e^4/8) + (11e^6/256) + (31e^8/1024)$$

$$c_2 = (3e^4/16) + (5e^6/64) + (25e^8/1024)$$

$$c_3 = (77e^6/768) + (59e^8/1024),$$

$$c_4 = 127e^8/2048$$

The distance h is given by

$$h/a = \cos\Delta\phi - a/N = \cos\Delta\phi - (1 - e^2\sin^2\phi)^{1/2}$$

$$= (1-e^2\sin^2\phi)^{-1/2}\left\{[1-e^2\sin^2\phi(1 + e^2\cos^2\phi)]^{1/2} - 1 + e^2\sin^2\phi\right\}$$

$$h = a(d_1 - d_2\cos 2\phi + d_3\cos 4\phi - d_4\cos 6\phi + d_5\cos 8\phi)$$

$$d_1 = (e^2/4) - (e^4/64) - (3e^6/256) - (233e^8/16384)$$

$$d_2 = (e^2/4) + (e^4/16) + (7e^6/512) + (3e^8/2048)$$

$$d_3 = (5e^4/64) + (11e^6/256) + (115e^8/4096)$$

$$d_4 = (9e^6/512) + (37e^8/2048)$$

$$d_5 = 53e^8/16384$$

STANDARD LATITUDE FORMULAS

The three latitudes usually associated with the auxiliary sphere ellipsoid configuration as shown in Figure 1, are the geocentric, parametric, and geodetic represented here by ψ, θ, and ϕ_0 respectively and related through the equations

$$\tan\psi/\tan\theta = \tan\theta/\tan\phi_0 = (1-e^2)^{1/2},$$

where e is the eccentricity of the meridian ellipse. The parametric latitude, θ, is also called here the geocentric latitude of points on the auxiliary sphere.

LATITUDES FOR CLARKE 1886 SPHEROID

Series representations, accurate to 0.001 second, for the differences in ϕ, ϕ_0, θ, ψ are:

$$\Delta\phi \text{ (seconds)} = \phi - \theta = 699''.2540 \sin 2\phi - 0''.5936 \sin 4\phi + 0''.0004 \sin 6\phi$$

$$\Delta\phi \text{ (seconds)} = \phi - \theta = 699''.2520 \sin 2\theta + 1''.7769 \sin 4\theta + 0''.0064 \sin 6\theta$$

$$\Delta\theta_0 \text{ (seconds)} = \phi - \phi_0 = 349''.0318 \sin 2\theta + 1''.4796 \sin 4\theta + 0''.0061 \sin 6\theta$$

$$h \text{ (meters)} = 10{,}788.3852 - 10{,}811.2646 \cos 2\phi + 22.9147 \cos 4\phi - 0.0350 \cos 6\phi$$

$$\phi_0 - \psi = 700''.4385 \sin 2\phi_0 - 1''.1893 \sin 4\phi_0 + 0''.0027 \sin 6\phi_0$$

$$\phi_0 - \psi = 700''.4385 \sin 2\psi + 1''.1893 \sin 4\psi + 0''.0027 \sin 6\psi$$

$$\phi_0 - \theta = 350''.2202 \sin 2\phi_0 - 0''.2973 \sin 4\phi_0 + 0''.0003 \sin 6\phi_0$$

$$\phi_0 - \theta = 350''.2202 \sin 2\theta + 0''.2973 \sin 4\theta + 0''.0003 \sin 6\theta$$

$$\theta - \psi = 350''.2202 \sin 2\theta - 0''.2973 \sin 4\theta + 0''.0003 \sin 6\theta$$

$$\theta - \psi = 350''.2202 \sin 2\psi + 0''.2973 \sin 4\psi + 0''.0003 \sin 6\psi$$

GREAT CIRCLE TRACK FORMULAS

First compute λ_0 and θ_0 from

$$\tan\lambda_0 = \frac{\tan\theta_2 \cos\lambda_1 - \tan\theta_1 \cos\lambda_2}{\tan\theta_1 \sin\lambda_2 - \tan\theta_2 \sin\lambda_1}$$

$\cot\theta_0 = \cot\theta_1 \cos(\lambda_0 - \lambda_1) = \cot\theta_2 \cos(\lambda_0 - \lambda_2)$. (See Figure 2).

Then compute α_1 and α_2 from

$$\sin\alpha_1 = \frac{\cos\theta_0}{\cos\theta_1}, \quad \sin\alpha_2 = \frac{\cos\theta_0}{\cos\theta_2}$$

Next compute S_1 and S_2 from

$$\tan S_1 = \cos\alpha_1 \cot\theta_1, \quad \tan S_2 = \cos\alpha_2 \cot\theta_2$$

The computations for α_1, α_2, S_1 and S_2 are checked by

$$\cos(\lambda_2 - \lambda_1) = \cos\alpha_1 \cos\alpha_2 + \sin\alpha_1 \sin\alpha_2 \cos(S_1 - S_2)$$

For equally spaced intervals along the great circle track, for instance in 100 nautical mile intervals, let $S = S_1 \pm 100K$, $K = 1, 2, 3, ---, n$. With these values of S one computes successively corresponding values of θ', λ', and α' from

$\sin\theta' = \sin\theta_0 \cos S$, $\tan(\lambda_0 - \lambda') = \tan S/\cos\theta_0$, $\tan\alpha' = \cot\theta_0 / \sin S$

and checks by means of $\sin\theta' \cdot \tan(\lambda_0 - \lambda') \cdot \tan\alpha' = 1$.

PARALLELS AT A GIVEN DISTANCE FROM THE GREAT CIRCLE TRACK

To compute the coordinates $(\theta p, \lambda p)$ and $(\theta p', \lambda p')$ of points at a given distance s from a given great circle track and symmetric with respect to it one computes (see Figure 3):

$\sin\theta_k = A\cos S \pm B$ when $k = p$, use + sign

$k = p'$, use − sign

$\sin(\lambda_0 - \lambda_k) = C\sin S/\cos\theta_k$

S and θ_0 are the same as given under the great circle track formulas above and $A = C\sin\theta_0$, $B = \cos\theta_0 \sin s$, $C = \cos s$. The computations may be checked by

$\cos 2s = \sin\theta p \sin\theta p' + \cos\theta p \cos\theta p' \cos(\lambda p' - \lambda p)$.

SPHERICAL RECTANGULAR COORDINATE SYSTEM WITH A GREAT CIRCLE BASE LINE AS AN AXIS

It is assumed that the base line has been established, that is the coordinates (θ_0, λ_0) of the vertex of the great circle base line have been computed from the coordinates of two given points $Q_1(\theta_1, \lambda_1)$, $Q_2(\theta_2, \lambda_2)$, see Figures 2 and 5.

Formulas for computing y, S, x from θ and λ

$$\sin y = \cos\theta_0 \sin\theta - \sin\theta_0 \cos\theta \cos(\lambda_0 - \lambda)$$

$$\tan S = \frac{\sin\theta_0 \cos\theta \sin(\lambda_0 - \lambda)}{\sin\theta - \cos\theta_0 \sin y} = \frac{\cos\theta \sin(\lambda_0 - \lambda)}{\sin\theta_0 \sin\theta + \cos\theta_0 \cos\theta \cos(\lambda_0 - \lambda)}$$

$$\sin x = \sin(S - S_1)\cos y$$

Formulas for computing S, θ, λ from x and y

Let $C = \cos y$, $D = \sin y$, $E = \sin x$, $A = C\sin\theta_0$, $B = D\cos\theta_0$, then

$$S = \arcsin(E/C) + S_1$$

$$\theta = \arcsin(A\cos S + B)$$

$$\lambda = \lambda_0 - \arcsin(C\sin S/\cos\theta)$$

SPHERICAL HYPERBOLA FORMULAS AND PLANE EQUIVALENTS

	Spherical	Plane
(1)	$\tan^2 r = \dfrac{\tan^2 a\,(\sin^2 c - \sin^2 a)}{\sin^2 c \cos^2 \alpha - \sin^2 a}$	$r^2 = \dfrac{a^2 (c^2 - a^2)}{c^2 \cos^2 \alpha - a^2}$
(2)	$\sin^2 x = \dfrac{\sin^2 a \cos^2 c}{\sin^2 c - \sin^2 a} \sin^2 y + \sin^2 a$	$x^2 = \dfrac{a^2 y^2}{c^2 - a^2} + a^2$
(3)	$\tan R = \dfrac{\cos 2c \pm \cos 2a}{\sin 2c \cos \beta \pm \sin 2a}$	$R = \dfrac{a^2 - c^2}{c \cos \beta - a}$
(4)	$\tan^2(\beta/2) = \dfrac{\sin(c-a)\sin(R+c+a)}{\sin(c+a)\sin(R-c+a)}$	$\tan^2(\beta/2) = \dfrac{(c-a)(R+c+a)}{(c+a)(R-c+a)}$

In (1) and (2) the origin of coordinates is the midpoint of $Q_1 Q_2$, see Figure 5. Equations (3) and (4) are two polar forms with origin at a focus Q_1, see Figures (5) and (6). Appendix 1 has computations based on the plane equivalent of (3).

DISTANCE AND AZIMUTH FORMULAS

Normal section azimuths (Geodetic latitude, ϕ)

$$\cot \alpha_{AB} = \frac{\left[\sin \phi_2 - (N_1/N_2)\sin \phi_1\right] e^2 \cos \phi_1 \sec \phi_2 + (\sin \phi_1 \cos \Delta\lambda - \tan \phi_2 \cos \phi_1)}{\sin \Delta\lambda}$$

$$\cot \alpha_{BA} = -\frac{\left[\sin \phi_1 - (N_2/N_1)\sin \phi_2\right] e^2 \cos \phi_2 \sec \phi_1 + (\sin \phi_2 \cos \Delta\lambda - \tan \phi_1 \cos \phi_2)}{\sin \Delta\lambda}$$

Normal Section Azimuths (parametric latitude θ)

$$\cot \alpha_{AB} = \frac{\sin \theta_1 \cos \Delta\lambda - \cos \theta_1 \tan \theta_2 + e^2 (\sin \theta_2 - \sin \theta_1) \cos \theta_1 \sec \theta_2}{(1 - e^2 \cos^2 \theta_1)^{1/2} \sin \Delta\lambda}$$

$$\cot \alpha_{BA} = -\frac{\sin \theta_2 \cos \Delta\lambda - \cos \theta_2 \tan \theta_1 + e^2 (\sin \theta_1 - \sin \theta_2) \cos \theta_2 \sec \theta_1}{(1 - e^2 \cos^2 \theta_2)^{1/2} \sin \Delta\lambda}$$

Great Elliptic Section Azimuths (Geodetic latitude ϕ)

$$\cot \alpha_{AB} = (1 - e^2)\frac{N_1^2}{a^2} \frac{(\tan \phi_1 \cos \Delta\lambda - \tan \phi_2) \cos \phi_1}{\sin \Delta\lambda}$$

$$\cot \alpha_{BA} = (1 - e^2)\frac{N_2^2}{a^2} \frac{(\tan \phi_1 - \tan \phi_2 \cos \Delta\lambda) \cos \phi_2}{\sin \Delta\lambda}$$

Great Elliptic Section Azimuths (parametric latitude θ)

$$\cot \alpha_{AB} = \frac{(\tan \theta_1 \cos \Delta\lambda - \tan \theta_2)(\cos \theta_1)(1 - e^2 \cos^2 \theta_1)^{1/2}}{\sin \Delta\lambda}$$

$$\cot \alpha_{BA} = \frac{(\tan \theta_1 - \tan \theta_2 \cos \Delta\lambda)(\cos \theta_2)(1 - e^2 \cos^2 \theta_2)^{1/2}}{\sin \Delta\lambda}$$

Great Elliptic Arc Distance

$$s/a = (d_1 + d_2) - \tfrac{1}{4} k^2 [(d_1 + d_2) - \sin (d_1 + d_2) \cos (d_1 - d_2)]$$
$$- (1/128) k^4 [6(d_1 + d_2) - 8 \sin (d_1 + d_2) \cos (d_1 - d_2) + \sin 2(d_1 + d_2) \cos 2(d_1 - d_2)]$$
$$- (1/1536) k^6 [30(d_1 + d_2) - 45 \sin (d_1 + d_2) \cos (d_1 - d_2) + 9 \sin 2(d_1 + d_2) \cos 2(d_1 - d_2)$$
$$- \sin 3(d_1 + d_2) \cos 3(d_1 - d_2)]$$

Where in terms of geodetic latitude ϕ,

$k = (e\sqrt{1 - e^2}/a) N_0 \sin \phi_0$, $d_1 = \arccos (N_1 \sin \phi_1/N_0 \sin \phi_0)$,

$d_2 = \arccos (N_2 \sin \phi_2/N_0 \sin \phi_0)$

$\sin \phi_0 = [J/(J + \sin^2 \Delta\lambda)]^{1/2}$, $J = \tan^2\phi_1 + \tan^2\phi_2 - 2 \tan \phi_1 \tan \phi_2 \cos \Delta\lambda$,

and in terms of parametric latitude θ

$k = e \sin \theta_0$, $d_1 = \arccos (\sin \theta_1/\sin \theta_0)$, $d_2 = \arccos (\sin \theta_2/\sin \theta_0)$

$\sin \theta_0 = [F/(F + \sin^2 \Delta\lambda)]^{1/2}$, $F = \tan^2\theta_1 + \tan^2\theta_2 - 2 \tan \theta_1 \tan \theta_2 \cos \Delta\lambda$.

Also in terms of parametric latitude θ, great ellipticarc distance

$$s = a \begin{bmatrix} d - (e^2/8) (Xd - Y \sin d) \\ - (e^4/512) [(6d - \sin 2d) X^2 - 8 (\sin d) XY + 2 (\sin 2d) Y^2] \\ - (e^6/12288) [3(10d - 3 \sin 2d) X^3 - 3(15 \sin d - \sin 3d) X^2Y + 18(\sin 2d) XY^2 - 4(\sin 3d) Y^3] \end{bmatrix}$$

where $X = \dfrac{(\sin \theta_1 + \sin \theta_2)^2}{1 + \cos d} + \dfrac{(\sin \theta_1 - \sin \theta_2)^2}{1 - \cos d}$,

$Y = \dfrac{(\sin \theta_1 + \sin \theta_2)^2}{1 + \cos d} - \dfrac{(\sin \theta_1 - \sin \theta_2)^2}{1 - \cos d}$, $d = d_2 - d_1$, where d_1, d_2 are spherical distances from $P_1(\theta_1, \lambda_1)$, $P_2(\theta_2, \lambda_2)$ to the vertex $P_0(\theta_0, \lambda_0)$.

NOTE: If $e^2 = 2f$, the higher order terms in f then ignored, this becomes the so-called Andoyer-Lambert approximation in terms of parametric latitude.

GEODESIC IN TERMS OF GREAT ELLIPTIC ARC, IN GEODETIC LATITUDE WITH SECOND ORDER TERMS IN THE FLATTENING

Given the points $P_1(\phi_1, \lambda_1)$, $P_2(\phi_2, \lambda_2)$ on the reference ellipsoid, P_2 west of P_1, west longitudes considered positive.

With $\phi_m = \tfrac{1}{2}(\phi_1 + \phi_2)$, $\Delta\phi_m = \tfrac{1}{2}(\phi_2 - \phi_1)$, $\Delta\lambda = \lambda_2 - \lambda_1$, $\Delta\lambda_m = \tfrac{1}{2}\Delta\lambda$,

Let $k = \sin \phi_m \cos \Delta\phi_m$, $K = \sin \Delta\phi_m \cos \phi_m$,

$H = \cos^2\Delta\phi_m - \sin^2\phi_m = \cos^2\phi_m - \sin^2\Delta\phi_m$

$L = \sin^2\Delta\phi_m + H \sin^2\Delta\lambda_m = \sin^2(d/2)$, $1 - L = \cos^2(d/2)$, $\cos d = 1 - 2L$,

$t = \sin^2 d = 4L(1 - L)$, $U = 2k^2/(1 - L)$, $V = 2K^2/L$; $X = U + V$, $Y = U - V$,

$T = d/\sin d = 1 + (t/6) + 3(t^2/40) + 5(t^3/112) + 35(t^4/1152) + 63(t^5/2816) +$, (1 radian = 206,264.8062 seconds

$E = 30 \cos d$, $A = 4T\,(8 + TE/15)$, $D = 4(6 + T^2)$, $B = -2D$,

$C = T - \tfrac{1}{2}(A + E)$, $f/4 = 0.000847518825$, $f^2/64 = 0.179572039 \times 10^{-6}$ (Clarke 1866)

$S = a \sin d\,[T - (f/4)\,(TX - 3Y) + (f^2/64)\{X(A + CX) + Y(B + EY) + DXY\}]$,

$\sin(\alpha_2 + \alpha_1) = (K \sin \Delta\lambda)/L$, $\sin(\alpha_2 - \alpha_1) = (k \sin \Delta\lambda)/(1 - L)$,

$\tfrac{1}{2}(\delta\alpha_2 + \delta\alpha_1) = -(f/2)\,H\,(T + 1)\sin(\alpha_2 + \alpha_1)$, $\tfrac{1}{2}(\delta\alpha_2 - \delta\alpha_1) = -(f/2)\,H\,(T - 1)\sin(\alpha_2 - \alpha_1)$,

$\alpha_{1-2} = \alpha_1 + \delta\alpha_1$, $\alpha_{2-1} = \alpha_2 + \delta\alpha_2$.

Additional check formulae

$$X = \frac{(\sin\phi_1 + \sin\phi_2)^2}{1 + \cos d} + \frac{(\sin\phi_1 - \sin\phi_2)^2}{1 - \cos d} = 2\sin^2\phi_0 = 2F/(F + \sin^2\Delta\lambda)$$

$$Y = \frac{(\sin\phi_1 + \sin\phi_2)^2}{1 + \cos d} - \frac{(\sin\phi_1 - \sin\phi_2)^2}{1 - \cos d} = 2\sin^2\phi_0 \cos(d_1 + d_2)$$

$$F = \tan^2\phi_1 + \tan^2\phi_2 - 2\tan\phi_1\tan\phi_2\cos\Delta\lambda$$

$\cos(d_1 + d_2) = Y/X$, $1 + \cos d = 8k^2/(X + Y)$, $1 - \cos d = 8K^2/(X - Y)$,

$$\cos d = 4\left(\frac{k^2}{X+Y} - \frac{K^2}{X-Y}\right),\quad 4\left(\frac{k^2}{X+Y} + \frac{K^2}{X-Y}\right) = 1.$$

NOTE: If the second order term is ignored, the resulting equations are the equivalent of the so called Andoyer-Lambert approximation in terms of geodetic latitude.

The quantities H, T, L, k, K enter into both distance and azimuth formulas. Distances are given within a meter and azimuths within a second over all lines in all latitudes and azimuths. Other advantages are (1) no conversion to parametric latitudes, (2) no square root calculations, (3) for desk computers the only tabular data required is a table of the natural trigonometric functions as Peter's eight place tables. (4) the formulas are adaptable to high speed computers. See Table 4 page 81 and Appendix 3, lines 12 through 16, for desk computer sample computations based on these formulas as checked against 5 Coast and Geodetic Survey specially computed lines. The mean difference for the 5 lines between true geodetic lengths and computed values was 0.15 meter with a maximum difference of 0.24 meter. The mean difference between true and computed azimuths was 0.59 second with a maximum difference of 0.93 second.

GEODESIC IN TERMS OF GREAT ELLIPTIC ARC, IN PARAMETRIC LATITUDE WITH SECOND ORDER TERMS IN THE FLATTENING

Given on the reference ellipsoid the points $P_1(\theta_1, \lambda_1)$, $P_2(\theta_2, \lambda_2)$; P_2 west of P_1, west longitudes considered positive. (Geodetic latitudes are converted to parametric by the relation $\tan\theta = (1 - f)\tan\phi$ or an equivalent formula). With $\theta_m = \tfrac{1}{2}(\theta_2 + \theta_1)$, $\Delta\theta_m = \tfrac{1}{2}(\theta_2 - \theta_1)$, $\Delta\lambda = \lambda_2 - \lambda_1$, $\Delta\lambda_m = \Delta\lambda/2$;

let $k = \sin\theta_m \cos\Delta\theta_m$, $K = \sin\Delta\theta_m \cos\theta_m$,

$H = \cos^2\Delta\theta_m - \sin^2\theta_m = \cos^2\theta_m - \sin^2\Delta\theta_m$,

$L = \sin^2\Delta\theta_m + H\sin^2\Delta\lambda_m = \sin^2 d/2$, $1-L = \cos^2 d/2$,

$\cos d = 1 - 2L$, $h = \sin^2 d = 4L(1-L)$, $U = 2k^2/(1-L)$,

$V = 2K^2/L$, $X = U + V$, $Y = U - V$,

$T = d/\sin d = 1 + (1/6)h + (3/40)h^2 + (5/112)h^3 + (35/1152)h^4 + (63/2816)h^5 + \cdots$,

$E_0 = -2\cos d$, $D_0 = 4T^2$, $A_0 = -D_0E_0$, $B_0 = -2D_0$, $C_0 = T - \tfrac{1}{2}(A_0 + E_0)$,

$S = a\sin d\,[T - (f/4)(TX - Y) + (f^2/64)(A_0X + B_0Y + C_0X^2 + D_0XY + E_0Y^2)]$

$\sin(\alpha_2 + \alpha_1) = (K\sin\Delta\lambda)/L$, $\sin(\alpha_2 - \alpha_1) = (k\sin\Delta\lambda)/(1-L)$

$\tfrac{1}{2}(\delta\alpha_2 + \delta\alpha_1) = -(f/2)\,TH\sin(\alpha_1 + \alpha_2)$

$\tfrac{1}{2}(\delta\alpha_2 - \delta\alpha_1) = -(f/2)\,TH\sin(\alpha_2 - \alpha_1)$

$\alpha_{1-2} = \alpha_1 + \delta\alpha_1$, $\alpha_{2-1} = \alpha_2 + \delta\alpha_2$

Additional check formulae

$$X = \frac{(\sin\theta_1 + \sin\theta_2)^2}{1 + \cos d} + \frac{(\sin\theta_1 - \sin\theta_2)^2}{1 - \cos d} = 2\sin^2\theta_0 = 2F/(F + \sin^2\Delta\lambda)$$

$$Y = \frac{(\sin\theta_1 + \sin\theta_2)^2}{1 + \cos d} - \frac{(\sin\theta_1 - \sin\theta_2)^2}{1 - \cos d} = 2\sin^2\theta_0\cos(d_1 + d_2)$$

$F = \tan^2\theta_1 + \tan^2\theta_2 - 2\tan\theta_1\tan\theta_2\cos\Delta\lambda$

$\cos(d_1 + d_2) = Y/X$, $1 + \cos d = 8k^2/(X + Y)$, $1 - \cos d = 8K^2/(X - Y)$,

$$\cos d = 4\left(\frac{k^2}{X+Y} - \frac{K^2}{X-Y}\right), \quad 4\left(\frac{k^2}{X+Y} + \frac{K^2}{X-Y}\right) = 1.$$

NOTE: If the second order term is ignored, the resulting equations are the equivalent of the so-called Andoyer-Lambert approximation in terms of parametric latitude.

TRANSFORMATIONS: GEODETIC TO PARAMETRIC — PARAMETRIC TO GEODETIC

If primed quantities denote those in geodetic latitude, then the transformation equations are:

$d' = d - (f/2)Y\sin d + (f^2/16)[4Y(X-3)\sin d + (2Y^2 - X^2)\sin 2d]$,

$\sin d' = \sin d - (f/4)Y\sin 2d$

$X' = X[1 + f(2 - X)]$

$Y' = Y[1 + f(2 - X)] + (f/2)(X^2 - Y^2)\cos d$

$d = d' + (f/2)Y'\sin d' + (f^2/16)[4Y'(X'-1)\sin d' + (2Y'^2 - X'^2)\sin 2d']$

$\sin d = \sin d' + (f/4)Y'\sin 2d'$

$X = X'[1 - f(2 - X')]$

$Y = Y'[1 - f(2 - X')] - (f/2)(X'^2 - Y'^2)\cos d'$

DIFFERENCE FORMULAS TO SECOND ORDER IN THE FLATTENING

$$d' - d = -(f/2)\,Y \sin d + (f^2/16)\,[4Y\,(X-3)\sin d + (2Y^2 - X^2)\sin 2d],$$

$$= -(f/2)\,Y' \sin d' - (f^2/16)\,[4Y'(X'-1)\sin d' + (2Y'^2 - X'^2)\sin 2d'];$$

$$X' - X = fX\,(2-X)\,\{1 + (f/2)\,(3-2X)\},$$

$$= fX'(2-X')\,\{1 - (f/2)\,(1-2X')\};$$

$$Y' - Y = fY(2-X) + (f/2)\,(X^2 - Y^2)\cos d + (f^2/8)\left[\begin{array}{l} 4Y\,(2-X)\,(3-2X) \\ +\,(X^2-Y^2)\,\{(11-5X)\cos d + Y\,(1-3\cos^2 d)\}\end{array}\right]$$

$$= fY'(2-X') + (f/2)\,(X'^2 - Y'^2)\cos d' - (f^2/8)\left[\begin{array}{l} 4Y'\,(2-X')\,(1-2X') \\ +\,(X'^2-Y'^2)\,\{2(5-3X')\cos d' + Y'(1-3\cos^2 d')\}\end{array}\right]$$

CHORD DISTANCE, c

$$c = a\left[\{1-\cos(d_1+d_2)\}\,\{2-k^2[1-\cos(d_1-d_2)]\}\right]^{1/2}$$

Where in terms of geodetic latitude ϕ,

$$d_1 = \arc\cos\,(N_1 \sin\phi_1 / N_0 \sin\phi_0),\quad d_2 = \arc\cos\,(N_2 \sin\phi_2 / N_0 \sin\phi_0)$$

$$k^2 = [e^2(1-e^2)/a^2]\,N_0^2 \sin^2\phi_0$$

in terms of parametric latitude θ

$$d_1 = \arc\cos\,(\sin\theta_1/\sin\theta_0),\quad d_2 = \arc\cos\,(\sin\theta_2/\sin\theta_0),\quad k^2 = e^2\sin^2\theta_0.$$

ANGLE OF DIP OF THE CHORD, β

$$\sin\beta = \left\{\frac{(1-e^2)\,[1-\cos(d_1+d_2)]}{[2-k^2\{1-\cos(d_1-d_2)\}]\,(1-e^2+k^2\cos^2 d_1)}\right\}^{1/2},$$

with k, d_1, d_2 expressible in terms of either geodetic or parametric latitude as given above.

MAXIMUM SEPARATION OF CHORD AND ELLIPTIC ARC, H_0

$$H_0 = \frac{2abo}{c}\sin \tfrac{1}{2}(d_1+d_2)\,[1-\cos\tfrac{1}{2}(d_1+d_2)],$$

where c is the chord length as given above, $bo = a\sqrt{1-k^2}$; c, k, d_1, d_2 expressible in either parametric or geodetic latitude as given above.

GEOGRAPHIC COORDINATES OF POINT OF MAXIMUM SEPARATION

$\tan\phi = R/D$, or $\cos 2\phi = (D^2 - R^2)/(D^2 + R^2)$, $\tan\lambda = (\cos\theta_2 \sin\Delta\lambda)/(\cos\theta_1 + \cos\theta_2\cos\Delta\lambda)$,

$R = \sin\theta_1 + \sin\theta_2$, $D = (0.996609925)\,(4\cos^2\tfrac{1}{2}d - R^2)^{1/2}$; d is spherical distance between the points $P_1(\theta_1, \lambda_1)$, $P_2(\theta_2, \lambda_2)$ on the ellipsoid, θ is parametric latitude, $\Delta\lambda = \lambda_2 - \lambda_1$. See Figure 23 for sample computation.

SECTION 1. LATITUDE FORMULAE

The auxiliary sphere, associated with an ellipsoid of reference, is the sphere tangent to the spheroid along the equator. If it is desired to work on this sphere with formulae for conversion to the spheroidal surface, then a correspondence between geocentric latitude θ on the sphere and geodetic latitude ϕ on the ellipsoid is needed. Longitudes will be the same.

Now there are three latitudes in geodetic usage associated with the auxiliary-sphere ellipsoid configuration as shown in Figure 1. The θ as shown, and which we shall call geocentric latitude, is called the reduced or parametric latitude since it is the eccentric angle of the meridian ellipse. The angle ψ , as shown, is called in geodetic nomenclature, the geocentric latitude since it is the angle measured from the center of the ellipsoid to the point R on the meridian from the equator. The angle ϕ_0, as shown, is a geodetic latitude corresponding to θ. The three latitudes ψ, θ, ϕ_0, are related through the equations

$$\tan \psi = \sqrt{1 - e^2} \tan \theta = (1 - e^2) \tan \phi_0 \qquad (1)$$

$$\text{or } \tan \psi / \tan \theta = \tan \theta / \tan \phi_0 = \sqrt{1 - e^2}.$$

where e is the eccentricity of the meridian ellipse [1].*

However, for working directly on the auxiliary sphere and transferring directly to the ellipsoid, if θ is the geocentric latitude of the point P(a cos θ, a sin θ) on the auxiliary sphere, then the latitude actually corresponding on the spheroid is that found by dropping a perpendicular upon the meridian ellipse from P meeting the meridian in Q as shown in Figure 1, the normal making the angle ϕ as shown with the equator. The distance PQ = h, and ϕ are needed for the conversion where $0 \leq h \leq a - b$, a and b the semimajor and semiminor axes of the spheroid. We now develop the necessary conversion formulas between ϕ and θ.

The law of sines applied to triangles POT, POK of figure 1, yields

$$\frac{Ne^2 \sin\phi}{\sin \Delta\phi} = \frac{h + N}{\cos \theta} = \frac{a}{\cos \phi}, \quad \frac{Ne^2 \cos\phi}{\sin \Delta\phi} = \frac{h + N(1-e^2)}{\sin \theta} = \frac{a}{\sin \phi}, \qquad (2)$$

where $N = a/\sqrt{1 - e^2 \sin^2\phi}$; e, a are the eccentricity and equatorial radius of the reference ellipsoid. $(\Delta\phi = \phi - \theta)$.

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*[1] Bracketed numbers refer to the list of references at the end of the section.

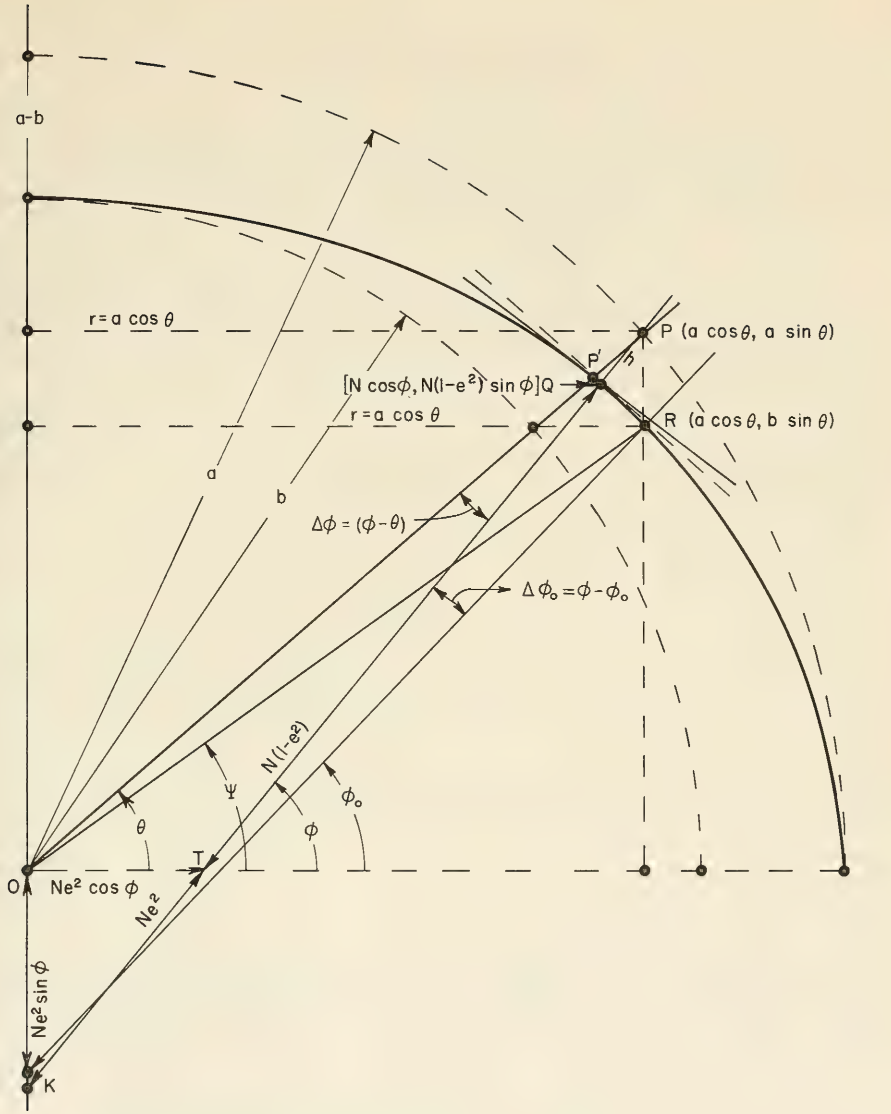

Figure 1. Latitude relationships in the auxiliary sphere-spheroid configuration.

From the first and last of either sets of equations (2) find

$$\sin \Delta\phi = \frac{e^2}{2a} \quad N \sin 2\phi = \frac{e^2 \sin\phi \cos \phi}{\sqrt{1 - e^2 \sin^2\phi}}. \tag{3}$$

To find the maximum value of $\Delta\phi$ and the value of ϕ at which the maximum occurs, one differentiates $\Delta\phi = \text{arc sin} \dfrac{e^2 \sin \phi \cos \phi}{\sqrt{1-e^2 \sin^2\phi}}$ to obtain

$$\frac{d\Delta\phi}{d\phi} = e^2 \frac{e^2 \cos^2 2\phi + 2(2 - e^2) \cos 2\phi + e^2}{(2 - e^2 + e^2 \cos 2\phi) \sqrt{2(2-e^2) - e^4 + 2e^2 \cos 2\phi + e^4 \cos^2 2\phi}}; \tag{4}$$

neither factor of the denominator of (4) is zero for $0 \leq \phi \leq 90°$. Hence to find the maximum from (4), place the numerator equal to zero and solve for $\cos 2\phi$ to obtain

$$\cos 2\phi = 1 + 2\left(\sqrt{1 - e^2} - 1\right)/e^2. \tag{5}$$

The flattening, f, of the reference ellipsoid is given by $f = (a-b)/a = 1 - b/a = 1 - \sqrt{1-e^2}$, whence $e^2 = 2f - f^2$, we can write

$$\cos 2\phi = 1 - 2(1 - \sqrt{1 - e^2})/e^2 = 1 - 2f/(2f - f^2) = -f/(2-f)$$

$$\sin^2 2\phi = 1 - \cos^2 2\phi = 1 - f^2/(2 - f)^2 = 4(1 - f)/(2 - f)^2$$

$$\sin^2\phi = \frac{1}{2} - \frac{1}{2} \cos 2\phi = \frac{1}{2} + \frac{f}{2(2-f)} = \frac{1}{2-f}.$$

$$1 - e^2 \sin^2\phi = 1 - f(2 - f)/(2 - f) = 1 - f.$$

from (3) $\quad \sin^2\Delta\phi = \dfrac{e^4}{4} \dfrac{\sin^2 2\phi}{1-e^2 \sin^2\phi} = \dfrac{f^2(2-f)^2}{4} \dfrac{4(1-f)}{(2-f)^2} \dfrac{1}{1-f}$

$$\sin^2 \Delta\phi = f^2$$

hence $\quad \sin \Delta\phi_{max} = f = 0.0033900753$ (Clarke 1866 ellipsoid).

$$\cos 2\phi = -0.001697914$$

$$\phi = 45° \, 02' 55''.106,$$

and $\quad \Delta\phi_{max} = 0° \, 11' \, 39''.255, \quad (6)$

$$\theta = \phi - \Delta\phi = 44° \, 51' \, 15''.851.$$

Now from (3) and $\theta = \phi - \Delta\phi$ a complete table for corresponding latitudes can be computed readily since complete tables for N to 0.001 meter have been computed for most reference ellipsoids. [2]

To develop $\sin \Delta\phi$ is a series for computation without the necessity of tables of N, write (3) in the form $\sin \Delta\phi = e^2 \sin \phi \cos \phi \cdot (1 - e^2 \sin^2 \phi)^{-1/2}$, then expand the radical by the binominal formula to get

$$\sin \Delta\phi = e^2 \sin \phi \cos \phi \left(1 + \frac{e^2}{2} \sin^2 \phi + \frac{3}{8} e^4 \sin^4 \phi + \frac{5}{16} e^6 \sin^6 \phi\right)$$

$$= \frac{e^2}{2} \sin 2\phi + \frac{e^4}{2} \sin^3\phi \cos\phi + \frac{3}{8} e^6 \sin^5\phi \cos\phi + \frac{5}{16} e^8 \sin^7\phi \cos\phi . \tag{7}$$

now $\sin^3\phi \cos\phi = \frac{1}{4} \sin 2\phi - \frac{1}{8} \sin 4\phi$

$$\sin^5\phi \cos\phi = \frac{5}{32} \sin 2\phi - \frac{1}{8} \sin 4\phi + \frac{1}{32} \sin 6\phi \tag{8}$$

$$\sin^7\phi \cos\phi = \frac{7}{64} \sin 2\phi - \frac{7}{64} \sin 4\phi + \frac{3}{64} \sin 6\phi - \frac{1}{128} \sin 8\phi,$$

and the values from (8) placed in (7) give

$$\sin \Delta\phi = c_1 \sin 2\phi - c_2 \sin 4\phi + c_3 \sin 6\phi - c_4 \sin 8\phi;$$

where $c_1 = \frac{e^2}{2} + \frac{e^4}{8} + \frac{15}{256} e^6 + \frac{35}{1024} e^8$, $c_2 = e^4/16 + \frac{3}{64} e^6 + \frac{35}{1024} e^8$, (9)

$c_3 = \frac{3}{256} e^6 + \frac{15}{1024} e^8$, $c_4 = \frac{5}{2048} e^8$

If $\Delta\phi$ in radians is desired rather than $\sin \Delta\phi$, then in the expansion

$$\text{arc} \sin x = x(1 + x^2/6 + - - -) \tag{10}$$

let $x = \sin \Delta\phi$, whence arc $\sin x = \Delta\phi$ and

$$\Delta\phi = \sin \Delta\phi \left(1 + \frac{\sin^2 \Delta\phi}{6} + - - - - \right). \tag{11}$$

from (9) with $e^2 = 0.006768657997$, find

$$c_1 = 0.003390074081,\ c_2 = 0.000002878029,$$
$$c_3 = 3.665 \times 10^{-9},\ c_4 = 5 \times 10^{-12} \text{ (negligible)}. \tag{12}$$

For estimation purposes the values in (12) may be written

$$c_1 = 3 \times 10^{-3},\ c_2 = 3 \times 10^{-6},\ c_3 = 4 \times 10^{-9}$$
$$c_1^2 = 9 \times 10^{-6},\ c_2^2 = 9 \times 10^{-12},\ c_3^2 = 2 \times 10^{-17}. \tag{13}$$

With the value of $\sin \Delta\phi$ from (9) in terms of the estimation coefficients (13) we examine the term $(\sin^3 \Delta\phi)/6$ in (11), and find that (11) may be written $\Delta\phi = \sin \Delta\phi +$

$$\frac{c_1^3}{6} \sin^3 2\phi - \frac{c_1^2 c_2}{2} \sin^2 2\phi \sin 4\phi. \tag{14}$$

since $\sin^3 2\phi = \frac{3}{4} \sin 2\phi - \frac{1}{4} \sin 6\phi$

$$\sin^2 2\phi \sin 4\phi = \frac{1}{2} \sin 4\phi - \frac{1}{4} \sin 8\phi, \tag{15}$$

equation (14) may be written, with the value of $\sin \Delta\phi$ from (9), as

$$\Delta\phi \text{(radians)} = \left((c_1 + \frac{c_1^3}{8})\right) \sin 2\phi - \left(c_2 + \frac{c_1^2 c_2}{4}\right) \sin 4\phi + \left(c_3 - \frac{c_1^3}{24}\right) \sin 6\phi, \tag{16}$$

or

$$\Delta\phi \text{ (seconds)} = (206{,}264.8062)\ \Delta\phi \text{ (radians)},$$

where c_1, c_2, c_3, are given by the expressions in (9) in terms of the eccentricity of the meridian ellipse.

We now check equations (9) and (17), using again values for the Clarke 1866 spheroid and for the maximum value of $\Delta\phi$.

From (9) and (12) we have

$$\sin \Delta\phi = 3.390074081 \times 10^{-3} \sin 2\phi - 2.878029 \times 10^{-6} \sin 4\phi + 3.665 \times 10^{-9} \sin 6\phi. \qquad (18)$$

From (12) and (17) find

$$\Delta\phi \text{ (seconds)} = 699''.2540 \sin 2\phi - 0''.5936 \sin 4\phi + 0''.0004 \sin 6\phi. \qquad (19)$$

Now with $\phi = 45° \, 02 \;\; 55''.106$ from (6), find $\sin 2\phi = +0.99999856$, $\sin 4\phi = -0.00339575$, $\sin 6\phi = -0.99998703$. (20)

The values from (20) placed in (18) give

$\sin \Delta\phi = 0.0033900753$ which checks the value found before in the 10th place. (See (6)).

The values from (20) placed in (19) give $\Delta\phi$ (seconds) $= 699''.2530 + ''.0020 - ''.0004 = 699''.2546$, or $11' \, 39''.255$ which is the value of $\Delta\phi_{max}$. (See (6)).

For explicit computation of ϕ as a function of θ, we obtain the following development. From the second and third of each set of equations (2), find

$h + N = a \cos\theta/\cos\phi = Ne^2 + a \sin\theta/\sin\phi$, whence

$$\tan\phi = \tan\theta + (e^2/a \cos\theta)(N \sin\phi)$$

$$\text{or} \quad \tan\phi = \tan\theta + \left(e^2\sqrt{1 + \tan^2\theta}\right)\left(\tan\phi/\sqrt{1 + (1 - e^2)\tan^2\phi}\right). \qquad (21)$$

(NOTE: Equation (21) also follows directly from (3) by expanding the left hand side and dividing every term by the product $\cos\phi \cos\theta$. $\sin \Delta\phi = \sin\phi \cos\theta - \cos\phi \sin\theta$.)

Now (21) is of the form

$$y = x + h(x)\, g(y)$$

and the Lagrange expansion formula may be used, [3].

Equation (21) may be written

$$y = x + \quad e^2(1 + x^2)^{1/2} \quad . \quad y[1 + (1 - e^2)\, y^2]^{-1/2} \qquad (22)$$

Where $y = \tan\phi$, $x = \tan\theta$, $h(x) = e^2(1 + x^2)^{1/2}$, $g(y) = y[1 + (1 - e^2)y^2]^{-1/2}$.

By use of the Lagrange expansion formula, a function f(y) which has a power series representation may be written

$$f(y) = f(x) + \sum_{n=1}^{\infty} \frac{\{h(x)\}^n}{n!} \frac{d^{\,n-1}}{dx^{\,n-1}} \; f'(x)\,\{g(x)\}^n \qquad (23)$$

With $y = \tan\phi$, $f(y) = \arctan y = \phi$; $x = \tan\theta$, $f(x) = \arctan x = \theta$, $f'(x) = \dfrac{1}{1 + x^2} = \cos^2\theta$,

equation (23) may be written

$$\Delta\phi = \phi - \theta = \sum_{n=1}^{\infty} \frac{e^{2n} \sec^n\theta}{n!} \frac{d^{\,n-1}}{dx^{\,n-1}}\, G(\theta) \qquad (24)$$

Where $G(\theta) = (\cos^2\theta)\left(\tan\theta/\sqrt{1 + (1 - e^2)\tan^2\theta}\right)^n$, $\theta = \arctan x$.

First write $G(\theta)$ in the form

$$G(\theta) = (\cos^2\theta)\,[\sin\theta(1 - e^2\sin^2\theta)^{-1/2}]^n. \tag{25}$$

We wish to retain terms to e^8, but no higher. Hence we expand the radical in (25) to powers of e^6 since for $n = 1$, equation (25) will be multiplied by e^2 as seen from (24). Using the binomial formula for the expansion we can write (25) as

$$G(\theta) = (\cos^2\theta)\,(\sin\theta + \tfrac{1}{2}e^2\sin^3\theta + (\tfrac{3}{8})\,e^4\sin^5\theta + (\tfrac{5}{16})\,e^6\sin^7\theta)^n. \tag{26}$$

To retain terms in e^8 we will need the first four terms of the expansion (24) and hence three derivatives of (26). Now $\theta = \arc\tan x$, $\dfrac{d\theta}{dx} = \dfrac{1}{1+x^2} = \cos^2\theta$, $\dfrac{d^2\theta}{dx^2} = -2\sin\theta\cos^3\theta$,

$$\frac{d^3\theta}{dx^3} = 2(3\sin^2\theta - \cos^2\theta)\cos^4\theta.$$

$$\frac{dG}{dx} = \frac{dG}{d\theta}\,\frac{d\theta}{dx} = \left(\frac{dG}{d\theta}\right)\cos^2\theta \tag{27}$$

$$\begin{aligned}\frac{d^2G}{dx^2} &= \left(\frac{d^2G}{d\theta^2}\right)\left(\frac{d\theta}{dx}\right)^2 + \left(\frac{dG}{d\theta}\right)\left(\frac{d^2\theta}{dx^2}\right)\\ &= \cos^3\theta\left[\left(\frac{d^2G}{d\theta^2}\right)\cos\theta - 2\left(\frac{dG}{d\theta}\right)\sin\theta\right]\end{aligned} \tag{28}$$

$$\begin{aligned}\frac{d^3G}{dx^3} &= \left(\frac{d^3G}{d\theta^3}\right)\left(\frac{d\theta}{dx}\right)^3 + 3\left(\frac{d^2G}{d\theta^2}\right)\left(\frac{d\theta}{dx}\right)\left(\frac{d^2\theta}{dx^2}\right) + \left(\frac{dG}{d\theta}\right)\left(\frac{d^3\theta}{dx^3}\right)\\ &= \cos^4\theta\left[\left(\frac{d^3G}{d\theta^3}\right)\cos^2\theta - 6\left(\frac{d^2G}{d\theta^2}\right)\cos\theta\sin\theta + 2\left(\frac{dG}{d\theta}\right)(3\sin^2\theta - \cos^2\theta)\right]\end{aligned} \tag{29}$$

Because of the factor e^{2n} as a multiplier in (24), we can assume the following terms for (26) for $n = 1, 2, 3, 4$:

n	$G(\theta)$	
1	$(\cos^2\theta)(\sin\theta + \tfrac{1}{2}e^2\sin^3\theta + (3/8)e^4\sin^5\theta + (5/16)e^6\sin^7\theta)$	(30)
2	$(\cos^2\theta)\,(\sin^2\theta + e^2\sin^4\theta + e^4\sin^6\theta)$	
3	$(\cos^2\theta)\,(\sin^3\theta + (3/2)e^2\sin^5\theta)$	
4	$(\cos^2\theta)\,(\sin^4\theta)$	

The terms of (24) are now formed by finding the derivatives of $G(\theta)$ with respect to θ using the appropriate form of $G(\theta)$ from (30) and finding

$\dfrac{dG}{dx}$, $\dfrac{d^2G}{dx^2}$, $\dfrac{d^3G}{dx^3}$ by means of (27), (28), and (29).

Thus it is found that the first four terms of (24) are

$e^2 \sin\theta\cos\theta + \tfrac{1}{2}e^4 \sin^3\theta\cos\theta + (3/8)e^6 \sin^5\theta\cos\theta + (5/16)e^8 \sin^7\theta\cos\theta;$

$e^4 \sin\theta\cos\theta + (2e^6 - 2e^4)\sin^3\theta\cos\theta + (3e^8 - 3e^6)\sin^5\theta\cos\theta - 4e^8\sin^7\theta\cos\theta;$

$e^6 \sin\theta\cos\theta + (5e^8 - {}^{35}\!/_6\, e^6)\sin^3\theta\cos\theta + ({}^{35}\!/_6\, e^6 - {}^{77}\!/_4\, e^8)\sin^5\theta\cos\theta + {}^{63}\!/_4\, e^8 \sin^7\theta\cos\theta;$

$e^8 \sin\theta\cos\theta - 12e^8 \sin^3\theta\cos\theta + 30e^8\sin^5\theta\cos\theta - 20e^8\sin^7\theta\cos\theta.$

Adding corresponding terms of these we have

$$\Delta\phi = \phi - \theta = (e^2 + e^4 + e^6 + e^8)\sin\theta\cos\theta - [(3/2)e^4 + (23/6)e^6 + 7e^8]\sin^3\theta\cos\theta \qquad (31)$$
$$+[(77/24)e^6 + (55/4)e^8]\sin^5\theta\cos\theta - (127/16)e^8\sin^7\theta\cos\theta.$$

Now $\sin\theta\cos\theta = \tfrac{1}{2}\sin 2\theta$

$$\sin^3\theta\cos\theta = \tfrac{1}{4}\sin 2\theta - (1/8)\sin 4\theta$$
$$\sin^5\theta\cos\theta = (5/32)\sin 2\theta - (1/8)\sin 4\theta + (1/32)\sin 6\theta \qquad (32)$$
$$\sin^7\theta\cos\theta = (7/64)\sin 2\theta - (7/64)\sin 4\theta + (3/64)\sin 6\theta - (1/128)\sin 8\theta.$$

The values from (32) placed in (31) give finally

$$\phi = \phi - \theta = C_1 \sin 2\theta + C_2 \sin 4\theta + C_3 \sin 6\theta + C_4 \sin 8\theta$$

where $C_1 = \tfrac{1}{2}e^2 + (1/8)e^4 + (11/256)e^6 + (31/1024)e^8$ (33)

$C_2 = (3/16)e^4 + (5/64)e^6 + (25/1024)e^8$

$C_3 = (77/768)e^6 + (59/1024)e^8$, $C_4 = (127/2048)e^8$.

Again for the Clarke 1866 spheroid

$e^2 = 0.006768657997$, $e^4 = 0.00004581473108$, (34)

$e^6 = 0.0000003101042459$, $e^8 = 0.000000002098989584$, whence from (33)

$C_1 = 3.390069228 \times 10^{-3}$, $C_2 = 8.614540216 \times 10^{-6}$, (35)

$C_3 = 3.12121 \times 10^{-8}$, $C_4 = 1.302 \times 10^{-10}$.

We now check (33) directly from the maximum value of $\Delta\phi$, the assumption being that if it holds for the maximum it will hold for all $\Delta\phi$.

From (6) $\theta = 44° 51' 15''.851$, whence

$\sin 2\theta = 0.99998708$, $\sin 4\theta = 0.01016441$, $\sin 6\theta = -0.99988377$, $\sin 8\theta = -0.02032777$. (36)

With the values from (35) and (36) find

$C_1 \sin 2\theta = 0.0033900254283$	$C_3 \sin 6\theta = -0.0000000312085$
$C_2 \sin 4\theta = 0.0000000875617$	$C_4 \sin 8\theta = -0.0000000000026$
0.0033901129900	-0.0000000312111

$\Delta\phi$ (radians) = 0.0033900817789

$\Delta\phi$ (seconds) = (0.0033900817789) (206,264.8062) = $699''.2545611$,

or $\Delta\phi_{max} = 11' \; 39''.255$ which checks (6).

Note that the term $C_4 \sin 8\theta$ does not contribute to the result. Also, only eight place tables of trigonometric natural functions were used, [4].

Hence for geodetic latitude ϕ corresponding to geocentric latitude θ on the auxiliary sphere, the following formulas are sufficient for any spheroid of reference to 0.001 second:

$$\Delta\phi \text{ (seconds)} = \phi - \theta = (206{,}264.8062)\,(C_1 \sin 2\theta + C_2 \sin 4\theta + C_3 \sin 6\theta)$$

$$C_1 = \tfrac{1}{2}e^2 + (1/8)e^4 + (11/256)e^6 + (31/1024)e^8,\quad C_2 = (3/16)e^4 + (5/64)e^6 + (25/1024)e^8, \tag{37}$$

$C_3 = (77/768)e^6 + (59/1024)e^8$, e is eccentricity of the meridian.

Now we have noted that the geocentric latitude θ as defined here is called the parametric or reduced latitude in geodetic nomenclature and has a corresponding geodetic latitude ϕ_0 as shown in Figure 1. From (1) we see that they are related by the equation $\tan \phi_0 = (\tan \theta)/\sqrt{1 - e^2}$. (38)
For instance from (6) for $\theta = 44^\circ\ 51'\ 15''.851$ find from (38) that $\phi_0 = 44^\circ\ 57'\ 06''.069$. Also from (6), $\phi = 45^\circ\ 02'\ 55''.106$, whence for $\theta = 44^\circ\ 51'\ 15''.851$ we have $\Delta\phi_0 = \phi - \phi_0 = 0^\circ\ 05'\ 49''.037$. (39)

Using the values from (34), equation (37) may be written for the Clarke 1866 spheroid as

$$\Delta\phi \text{ (seconds)} = \phi - \theta = 699''.2520 \sin 2\theta + 1''.7769 \sin 4\theta + 0''.0064 \sin 6\theta. \tag{40}$$

From C. & G.S. special publication No. 67, [5], find

$$\phi_0 - \theta = 350''.2202 \sin 2\theta + 0''.2973 \sin 4\theta + 0''.0003 \sin 6\theta. \tag{41}$$

Subtracting (41) from (40) one finds

$$\Delta\phi_0 = \phi - \phi_0 = 349''.0318 \sin 2\theta + 1''.4796 \sin 4\theta + 0''.0061 \sin 6\theta. \tag{42}$$

With $\theta = 44^\circ\ 51'\ 15''.851$ and the values from (28), equation (42) gives

$\Delta\phi_0 = 5'\ 49''.036$ which is within 0.001 second of (39).

From the second and third members of each set of equations (2) find

$$h = a \sin\theta \csc\phi - (1 - e^2)\,N = a \cos\theta \sec\phi - N. \tag{43}$$

To develop h in a power series in ϕ, free of N and θ, refer again to Figure 1. If the tangent at Q meets OP in P', then $PP' = a - (a^2/N) \sec \Delta\phi$, $h = PP' \cos \Delta\phi$, whence

$$h/a = \cos \Delta\phi - a/N = \cos \Delta\phi - \sqrt{1 - e^2 \sin^2\phi} \tag{44}$$

With $\cos \Delta\phi = \sqrt{1 - \sin^2 \Delta\phi}$, and the value of $\sin \Delta\phi$ from (3), (44) may be written

$$h/a = (1 - e^2 \sin^2\phi)^{-1/2} \{[1 - e^2 \sin^2\phi\,(1 + e^2 \cos^2\phi)]^{1/2} - 1 + e^2 \sin^2\phi\}. \tag{45}$$

The relation (45) may also be obtained directly from equation (2) by eliminating θ between the equations $a \cos\theta = (h + N) \cos\phi$ and $a \sin\theta = [h + N(1 - e^2)] \sin\phi$.

Expanding the two radicals by the binomial formula, (45) may be written

$$\begin{aligned} h/a &= (e^2/2 - e^4/2) \sin^2\phi + [(5/8)e^4 - \tfrac{1}{2}e^6 - (1/8)e^8] \sin^4\phi \\ &\quad + [(9/16)e^6 - (1/4)e^8] \sin^6\phi + (53/128)e^8 \sin^8\phi \end{aligned} \tag{46}$$

Now $\sin^2 \phi = \frac{1}{2}(1 - \cos 2\phi)$

$\sin^4 \phi = 3/8 - \frac{1}{2} \cos 2\phi + (1/8) \cos 4\phi$

$\sin^6 \phi = 5/16 - (15/32) \cos 2\phi + (3/16) \cos 4\phi - (1/32) \cos 6\phi$

$\sin^8 \phi = 35/128 - (7/16) \cos 2\phi + (7/32) \cos 4\phi - (1/16) \cos 6\phi + (1/128) \cos 8\phi$

and these values placed in (46) give

$$\begin{aligned} h &= a\,(d_1 - d_2 \cos 2\phi + d_3 \cos 4\phi - d_4 \cos 6\phi + d_5 \cos 8\phi) \\ d_1 &= e^2/4 - e^4/64 - (3/256)e^6 - (233/16{,}384)e^8, \\ d_2 &= e^2/4 + e^4/16 + 7e^6/512 + 3e^8/2048, \\ d_3 &= 5e^4/64 + 11e^6/256 + 115e^8/4096 \\ d_4 &= 9e^6/512 + 37e^8/2048,\quad d_5 = 53e^8/16{,}384 \end{aligned} \qquad 0 \le h \le a - b \tag{47}$$

a, e are the semimajor axis, eccentricity of the reference ellipsoid.

We now check (47) using the values of a and e for the Clarke 1866 spheroid. From (34) and (47) with a = 6,378,206.4 meters one has

$$h(\text{meters}) = 10{,}788.3852 - 10{,}811.2646 \cos 2\phi + 22.9147 \cos 4\phi - 0.0350 \cos 6\phi. \tag{48}$$

As a check, equation (48) should give

$h = a - b = 6{,}378{,}206.4 - 6{,}356{,}583.8 = 21{,}622.6$ meters

when $\phi = 90°$. Placing $\phi = 90°$ in (48) gives

$h = 10{,}788.3852 + 10{,}811.2646 + 22.9147 + 0.0350 = 21{,}622.5995$ meters.

Since we have the values of θ and ϕ for $\Delta\phi_{max}$ from (6) we now check the value given by (48) against the closed formula (43),

$$h = a \frac{\cos \theta}{\cos \phi} - N(\phi).$$

$\phi = 45° \, 02' \, 55''.106$, $\cos \phi = 0.70650624$, $\cos 2\phi = -0.00169788$

$\cos 4\phi = -0.99999423$, $\cos 6\phi = +0.00509360$.

$\theta = 44° \, 51' \, 15''.851$, $\cos \theta = 0.70890136$, $N(\phi) = 6{,}389{,}045.266$.

$$h = a \frac{\cos \theta}{\cos \phi} - N(\phi) = (6{,}378{,}206.4)\,(0.70890136)\,/\,(0.70650624) - 6{,}389{,}045.266$$

$$= 6{,}399{,}829.094 - 6{,}389{,}045.266 = 10{,}783.828 \text{ meters}$$

Equation (48) gives

$h = 10{,}788.3852 + 18.3562 - 22.9146 - 0.0002 = 10{,}783.827$ meters,

when $\phi = 0$, $h = 0$ and (48) gives

$h = 10{,}788.3852 - 10{,}811.2646 + 22.9147 - 0.0350 = + 0.0003$ meter.

Unless h were required to very high precision it is clear from the above checks that the formula (48) is adequate.

SUMMARY OF LATITUDE FORMULAE

If θ is the geocentric latitude of a point P (a cos θ, a sin θ) on the auxiliary sphere, then the corresponding geodetic latitude ϕ of P at an altitude h above the ellipsoid reference, as shown in figure 1, is given by

$$\sin \Delta\phi = \sin (\phi - \theta) = (e^2/2a)\, N \sin 2\phi = (e^2 \sin \phi \cos \phi)/\sqrt{1 - e^2 \sin^2\phi}$$

$$= c_1 \sin 2\phi - c_2 \sin 4\phi + c_3 \sin 6\phi - c_4 \sin 8\phi, \qquad (49)$$

$c_1 = e^2/2 + e^4/8 + 15e^6/256 + 35e^8/1024,$

$c_2 = e^4/16 + 3e^6/64 + 35e^8/1024$

$c_3 = 3e^6/256 + 15e^8/1024,\ c_4 = 5e^8/2048$

e = eccentricity of the meridian ellipse.

With the same coefficients as (49), we have

$$\Delta\phi \text{ (radians)} = (c_1 + c_1^3/8) \sin 2\phi - (c_2 + \frac{c_1^2}{4} c_2) \sin 4\phi + (c_3 - \frac{c_1^3}{24}) \sin 6\phi \qquad (50)$$

and in seconds

$$\Delta\phi\text{(seconds)} = (206{,}264.8062)\,[\,(c_1 + c_1^3/8) \sin 2\phi - (c_2 + c_1^2 c_2/4) \sin 4\phi + (c_3 - c_1^3/24) \sin 6\phi\,]. \qquad (51)$$

To express $\Delta\phi$ in terms of θ, instead of ϕ, we have the relation

$\tan \phi = \tan \theta + (e^2/a \cos \theta)\, N \sin \phi$

Which may be expanded by use of the Lagrange expansion formula to give

$\Delta\phi = \phi - \theta = C_1 \sin 2\theta + C_2 \sin 4\theta + C_3 \sin 6\theta + C_4 \sin 8\theta$

$C_1 = e^2/2 + e^4/8 + 11e^6/256 + 31e^8/1024, \qquad (52)$

$C_2 = 3e^4/16 + 5e^6/64 + 25e^8/1024,$

$C_3 = 77e^6/768 + 59e^8/1024,\ C_4 = 127e^8/2048.$

For checks within 0.001 second, (52) may be written $\Delta\phi$ (seconds) = (206,264.8062)

$$(C_1 \sin 2\theta + C_2 \sin 4\theta + C_3 \sin 6\theta) \qquad (53)$$

with C_1, C_2, C_3 the same as in (52).

$h/a = \cos \Delta\phi - a/N = (1 - e^2 \sin^2\phi)^{-1/2}\{[1 - e^2 \sin^2 \phi\, (1 + e^2 \cos^2 \phi)]^{1/2} - 1 + e^2 \sin^2 \phi\}$

$$h = a(d_1 - d_2 \cos 2\phi + d_3 \cos 4\phi - d_4 \cos 6\phi + d_5 \cos 8\phi) \qquad (54)$$

$d_1 = e^2/4 - e^4/64 - 3e^6/256 - 233e^8/16{,}384$

$d_2 = e^2/4 + e^4/16 + 7e^6/512 + 3e^8/2048 \qquad 0 \le h \le a-b$

$d_3 = 5e^4/64 + 11e^6/256 + 115e^8/4096$

$d_4 = 9e^6/512 + 37e^8/2048,\ d_5 = 53e^8/16{,}384$

a = radius of the auxiliary sphere (semimajor axis of the reference ellipsoid).

For the Clarke 1866 spheroid of reference we have from the above formulas:

$$\Delta\phi \text{ (seconds)} = \phi - \theta = 699''.2540 \sin 2\phi - 0''.5936 \sin 4\phi + 0''.0004 \sin 6\phi, \tag{55}$$

$$\Delta\phi \text{ (seconds)} = \phi - \theta = 699''.2520 \sin 2\theta + 1''.7769 \sin 4\theta + 0''.0064 \sin 6\theta, \tag{56}$$

$$\Delta\phi_0 \text{ (seconds)} = \phi - \phi_0 = 349''.0318 \sin 2\theta + 1''.4796 \sin 4\theta + 0''.0061 \sin 6\theta, \tag{57}$$

$$\text{h (meters)} = 10{,}788.3852 - 10{,}811.2646 \cos 2\phi + 22.9147 \cos 4\phi - 0.0350 \cos 6\phi. \tag{58}$$

For the Clarke 1866 spheroid, the maximum value of $\Delta\phi$ was found to be $11' \, 39''.255$ at $\phi = 45° \, 02' \, 55''.106$.

The value of $\Delta\phi_0$, at this maximum of $\Delta\phi$, was found to be $5' \, 49''.037$. Finally (58) was checked at $\phi = 0$, $90°$ and $\phi = 45° \, 02' \, 55''.106$. At $\phi = 90°$, the check was within 0.0005 meter; at $\phi = 0$, it was within 0.0003 meter; at $\phi = 45° \, 02' \, 55''.106$, it was within 0.001 meter.

The following latitude formulae are from C & G.S. Special Publication No. 67, [5], Where ϕ_0, ψ, θ are shown in figure 1.

$$\phi_0 - \psi = 700''.4385 \sin 2\phi_0 - 1''.1893 \sin 4\phi_0 + 0''.0027 \sin 6\phi_0 \tag{59}$$

$$\phi_0 - \psi = 700''.4385 \sin 2\psi + 1''.1893 \sin 4\psi + 0''.0027 \sin 6\psi \tag{60}$$

$$\phi_0 - \theta = 350''.2202 \sin 2\phi_0 - 0''.2973 \sin 4\phi_0 + 0''.0003 \sin 6\phi_0 \tag{61}$$

$$\phi_0 - \theta = 350''.2202 \sin 2\theta + 0''.2973 \sin 4\theta + 0''.0003 \sin 6\theta \tag{62}$$

$$\theta - \psi = 350''.2202 \sin 2\theta - 0''.2973 \sin 4\theta + 0''.0003 \sin 6\theta \tag{63}$$

$$\theta - \psi = 350''.2202 \sin 2\psi + 0''.2973 \sin 4\psi + 0''.0003 \sin 6\psi \tag{64}$$

The above are the series expansions for the expressions given as equation (1) page 12, that is

$$\tan \psi = \sqrt{1 - e^2} \, \tan \theta = (1 - e^2) \tan \phi_0. \tag{65}$$

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SECTION 2. SPHERICAL RECTANGULAR COORDINATE SYSTEM; LOCI

THE GREAT CIRCLE TRACK AS DETERMINED BY THE GEOGRAPHICAL COORDINATES OF TWO GIVEN POINTS ON THE AUXILIARY SPHERE

In figure 2, the two given points are $Q_1(\theta_1, \lambda_1)$, $Q_2(\theta_2, \lambda_2)$. The great circle track is then determined from the spherical triangle PQ_1Q_2. In order to simplify the computations and to have well balanced triangles from which to compute, one finds the point $O(\theta_0, \lambda_0)$ where the great circle Q_1Q_2 is orthogonal to a meridian λ_0. One then works from the right spherical triangle POQ' by adding or subtracting increments of distance from $S_1 = OQ_1$ to get the distance S. One always has then a strong right triangle POQ' from which to compute the latitude, longitude and azimuth α of the point $Q'(\theta', \lambda')$ on the base line Q_1Q_2.

DERIVATION OF FORMULAE

From right spherical triangle POQ'

$$\cos(\lambda_0 - \lambda') = \tan\left(\frac{\pi}{2} - \theta_0\right)\cot\left(\frac{\pi}{2} - \theta'\right) = \cot\theta_0 \tan\theta' \tag{1}$$

If the points Q_1 and Q_2 satisfy (1), we have by substituting their coordinates in (1)

$$\cos(\lambda_0 - \lambda_1) = \cot\theta_0 \tan\theta_1, \tag{2}$$

$$\cos(\lambda_0 - \lambda_2) = \cot\theta_0 \tan\theta_2$$

By forming the ratios of (2), expanding $\cos(\lambda_0 - \lambda_1)$ and $\cos(\lambda_0 - \lambda_2)$, dividing the left member numerator and denominator by $\cos\lambda_0$ one derives the formula

$$\tan\lambda_0 = \frac{\tan\theta_2 \cos\lambda_1 - \tan\theta_1 \cos\lambda_2}{\tan\theta_1 \sin\lambda_2 - \tan\theta_2 \sin\lambda_1} \tag{3}$$

Equations (2) may be written as

$$\cot\theta_0 = \cot\theta_1 \cos(\lambda_0 - \lambda_1) = \cot\theta_2 \cos(\lambda_0 - \lambda_2) \tag{4}$$

From right spherical triangle POQ' one has also

$$\sin\alpha' = \frac{\sin\left(\frac{\pi}{2} - \theta_0\right)}{\sin\left(\frac{\pi}{2} - \theta'\right)} = \frac{\cos\theta_0}{\cos\theta'}, \tag{5}$$

$$\cos\alpha' = \frac{\tan S}{\tan\left(\frac{\pi}{2} - \theta'\right)} = \tan S \tan\theta', \tag{6}$$

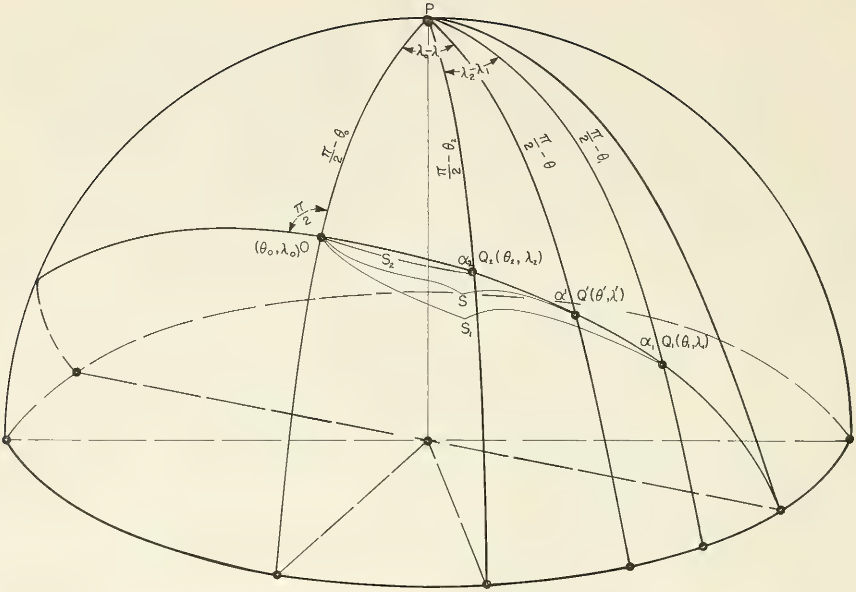


Figure 2. The great circle track configuration.

$$\sin \theta' = \cos S \sin \theta_0 , \tag{7}$$

$$\tan (\lambda_0 - \lambda') = \frac{\tan S}{\sin\left(\frac{\pi}{2} - \theta_0\right)} = \frac{\tan S}{\cos \theta_0} , \tag{8}$$

$$\tan \alpha' = \frac{\tan \left(\frac{\pi}{2} - \theta_0\right)}{\sin S} = \frac{\cot \theta_0}{\sin S} \tag{9}$$

$$\sin \theta' = \cot (\lambda_0 - \lambda') \cot \alpha' \text{ or}$$

$$\tan \alpha' \sin \theta' \tan (\lambda_0 - \lambda') = 1 \tag{10}$$

From the oblique spherical triangle PQ_1Q_2 find

$$\cos (\lambda_2 - \lambda_1) = - \cos (\pi - \alpha_2) \cos \alpha_1 + \sin (\pi - \alpha_2) \sin \alpha_1 \cos (S_1 - S_2) \quad \text{or}$$

$$\cos (\lambda_2 - \lambda_1) = \cos \alpha_1 \cos \alpha_2 + \sin \alpha_1 \sin \alpha_2 \cos (S_1 - S_2). \tag{10.1}$$

Computations from the formulae

First compute λ_0 and θ_0 from (3) and (4).

$$\tan \lambda_0 = \frac{\tan \theta_2 \cos \lambda_1 - \tan \theta_1 \cos \lambda_2}{\tan \theta_1 \sin \lambda_2 - \tan \theta_2 \sin \lambda_1}$$

$$\cot \theta_0 = \cot \theta_1 \cos (\lambda_0 - \lambda_1) = \cot \theta_2 \cos (\lambda_0 - \lambda_2)$$

Next compute α_1 and α_2 from (5),

$$\sin \alpha_1 = \frac{\cos \theta_0}{\cos \theta_1} , \quad \sin \alpha_2 = \frac{\cos \theta_0}{\cos \theta_2}$$

Then S_1 and S_2 from (6)

$$\tan S_1 = \cos \alpha_1 \cot \theta_1 , \quad \tan S_2 = \cos \alpha_2 \cot \theta_2$$

The computations for α_1, α_2; S_1 and S_2 are checked by (10.1)

$$\cos (\lambda_2 - \lambda_1) = \cos \alpha_1 \cos \alpha_2 + \sin \alpha_1 \sin \alpha_2 \cos (S_1 - S_2).$$

Now for equally spaced intervals along the great circle track, for instance in 100 nautical mile intervals, let $S = S_1 \pm 100k$.

$k = 1, 2, 3, \ldots\ldots. N.$

With these values of S one computes successively corresponding values of θ', λ' and α' from equations (7), (8), and (9)

$$\sin \theta' = \sin \theta_0 \cos S, \tan (\lambda_0 - \lambda') = \frac{\tan S}{\cos \theta_0} , \tan \alpha' = \frac{\cot \theta_0}{\sin S} .$$

These last computations are checked by (10)

$$\sin \theta' \cdot \tan (\lambda_0 - \lambda') \cdot \tan \alpha' = 1.$$

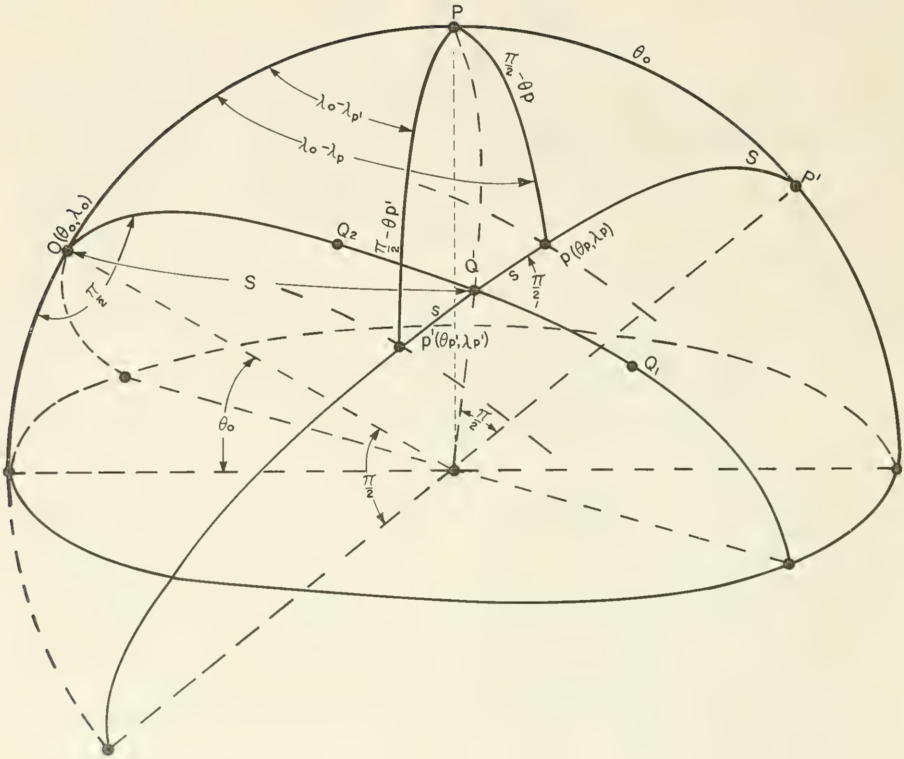


Figure 3. Parallels at a given distance from a great circle track.

PARALLELS AT A GIVEN DISTANCE FROM A GREAT CIRCLE TRACK

In Figure 3, the basic great circle track determined by $Q_1(\theta_1, \lambda_1)$, $Q_2(\theta_2, \lambda_2)$ is the same and the point $O(\theta_0, \lambda_0)$ is the same – (vertex of the great circle track). The point P' is the pole of the great circle determined by Q_1, Q_2. The angle at P' of the spherical triangle $P'PQ'$ is the distance $S = OQ'$ along the great circle track. If p and p' are points on the parallels at a distance s from the great circle track, then the coordinates of p and p' can be computed from the two spherical triangles $PP'p$, $PP'p'$, (Figure 4).

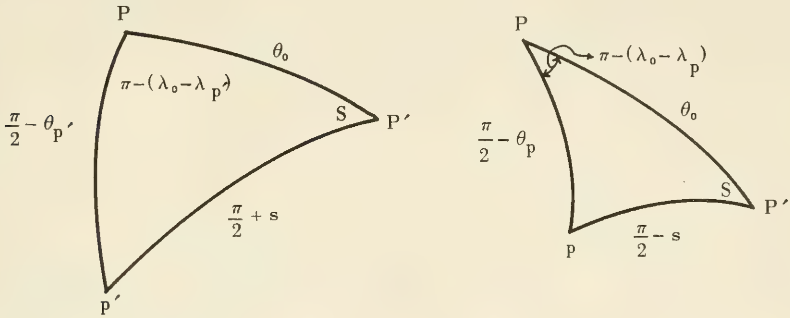


Figure 4

From these triangles one has

$$\sin\theta p = \cos\theta_0 \sin s + \sin\theta_0 \cos s \cos S$$

$$\sin\theta p' = -\cos\theta_0 \sin s + \sin\theta_0 \cos s \cos S \tag{11}$$

$$\frac{\cos s}{\sin(\lambda_0 - \lambda p)} = \frac{\cos\theta p}{\sin S}, \quad \frac{\cos s}{\sin(\lambda_0 - \lambda p')} = \frac{\cos\theta p'}{\sin S} \tag{12}$$

From (11) and (12) one may write

$$\sin\theta_k = A\cos S \pm B$$

$$\sin(\lambda_0 - \lambda_k) = C\sin S/\cos\theta_k \tag{13}$$

where $A = \sin\theta_0 \cos s$, $B = \cos\theta_0 \sin s$, $C = \cos s$.

A, B, C are constants for a given s. When $k = p$, the + sign is used in the first of equations (13). When $k = p'$, the – sign is used.

The computations may be checked as before by means of the equation

$$\cos 2s = \sin\theta p \sin\theta p' + \cos\theta p \cos\theta p' \cos(\lambda p' - \lambda p).$$

A SPHERICAL RECTANGULAR COORDINATE SYSTEM WITH A GREAT CIRCLE BASE LINE AS AN AXIS

Figure 5 is a further elaboration of Figures 2 and 3. M is the midpoint of the spherical segment Q_1Q_2. The section $MP'P''$ is perpendicular to the base line at M. The general point $Q(\theta, \lambda)$ has for the foot of the perpendicular from Q upon the base line, the point $Q'(\theta', \lambda')$ as shown in figure 2. The great circle arc QQ' passes through P_1' and QQ' is taken for spherical rectangular coordinate y. The great circle perpendicular to the section $MP'P''$ and passing through Q meets $MP'P''$ in T. The distance OQ' is S as shown in Figure 5. Note that the s of Figure 3 in the y of Figure 5. The great circle arc QT is taken for x. That is the spherical rectangular system chosen is $x = QT$, $y = QQ'$. Spherical polar coordinates are then r and α as shown in Figure 5, where $r = MQ$, and α is the angle between r and MQ'.

From the right spherical triangles MQT, MQQ' one finds

$$\sin x = \sin r \cos \alpha$$

$$\sin y = \sin r \sin \alpha \tag{14}$$

whence

$$\left.\begin{aligned} \sin r &= (\sin^2 x + \sin^2 y)^{1/2} \\ \tan \alpha &= \sin y / \sin x, \end{aligned}\right\} \tag{15}$$

that is (14) and (15) represent the conversion formulas between the spherical rectangular and spherical polar systems as given.

We now develop the coordinates x and y as functions of S and of θ and λ. Also θ and λ as functions of x and y.

COMPUTATION OF S, x, y, FROM θ AND λ

Assume that the base line has been established, that is the coordinates θ_0, λ_0 of the vertex, 0, of the great circle base line have been computed from the coordinates of the two given points $Q_1(\theta_1, \lambda_1)$, $Q_2(\theta_2, \lambda_2)$ by means of the equations as given on page 23. Then referring to Figure 5, find in spherical triangles:

PP'Q: $\cos y \sin S = \cos \theta \sin(\lambda_0 - \lambda)$, (16)

: $\sin y = \cos \theta_0 \sin \theta - \sin \theta_0 \cos \theta \cos(\lambda_0 - \lambda)$, (17)

OPQ: $\cos f = \sin \theta_0 \sin \theta + \cos \theta_0 \cos \theta \cos(\lambda_0 - \lambda)$, (18)

OQQ': $\cos y \cos S = \cos f$, (19)

TP'Q: $\sin x = \sin d \cos y$. (20)

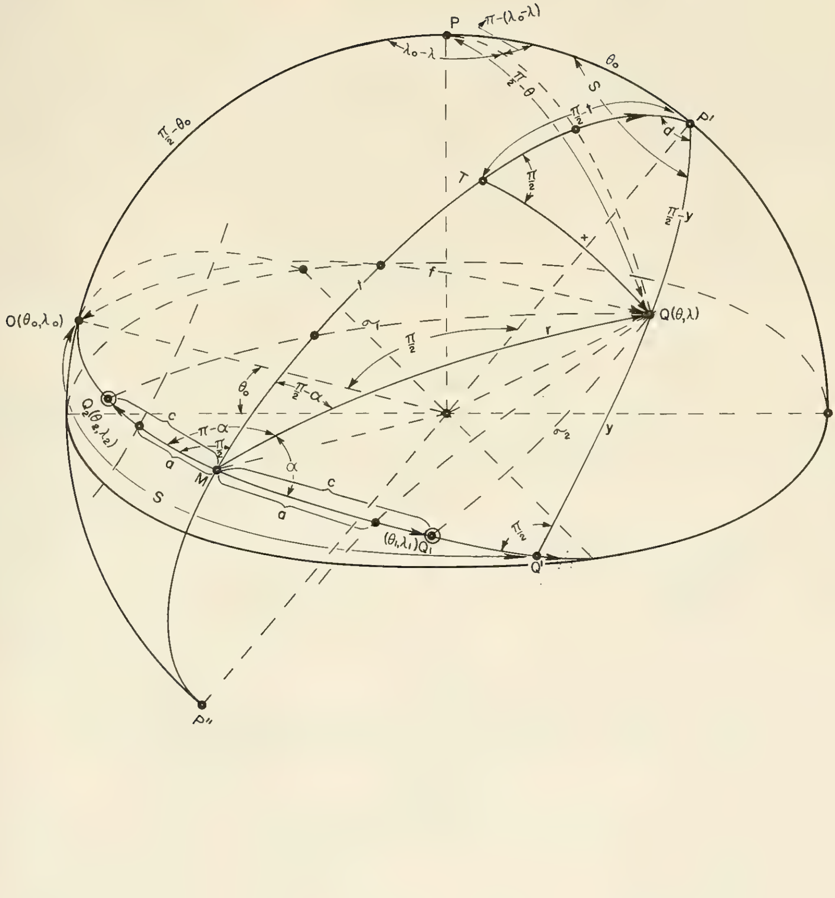


Figure 5. Spherical rectangular coordinate system.

Dividing respective members of (16) and (19) find

$$\tan S = \cos\theta \sin(\lambda_0 - \lambda) / \cos f \tag{21}$$

where cos f is given by (18).

From (17) and (18) we have $\sin\theta_0 \cos f = \sin\theta - \cos\theta_0 \sin y$ whence (21) may be written

$$\tan S = \frac{\sin\theta_0 \cos\theta \sin(\lambda_0 - \lambda)}{\sin\theta - \cos\theta_0 \sin y} \tag{22}$$

Referring now to Figures 1 and 5, it is seen that $d = MQ' = S - \frac{1}{2}(S_1 + S_2)$, where S_1 and S_2 are the distances from $O(\theta_0, \lambda_0)$ to Q_1 and Q_2 respectively.

Hence given the spherical curvilinear coordinates θ, λ of a point $Q(\theta, \lambda)$, to find S, x and y with θ_0, λ_0, S_1, S_2 known, compute y and S from (17) and (21) or (22) and then x from (20), i.e.

$$\sin y = \cos\theta_0 \sin\theta - \sin\theta_0 \cos\theta \cos(\lambda_0 - \lambda)$$

$$\tan S = \frac{\sin\theta_0 \cos\theta \sin(\lambda_0 - \lambda)}{\sin\theta - \cos\theta_0 \sin y} = \frac{\cos\theta \sin(\lambda_0 - \lambda)}{\cos f} \tag{23}$$

$$= \frac{\cos\theta \sin(\lambda_0 - \lambda)}{\sin\theta_0 \sin\theta + \cos\theta_0 \cos\theta \cos(\lambda_0 - \lambda)}$$

$$\sin x = \sin d \cos y = \sin\left[S - \tfrac{1}{2}(S_1 + S_2)\right] \; (1 - \sin^2 y)^{1/2}$$

COMPUTATION OF S, θ, λ FROM x AND y

From equation (20) one has $\sin d = \sin x / \cos y$ or $\sin[S - \frac{1}{2}(S_1 + S_2)] = \sin x / \cos y$ whence

$$S = \arcsin(\sin x / \cos y) + \tfrac{1}{2}(S_1 + S_2). \tag{24}$$

From equations (13) page 27,

$$\sin\theta = A \cos S + B \tag{25}$$

$$\sin(\lambda_0 - \lambda) = C \sin S / \cos\theta$$

where $A = C \sin\theta_0$, $B = D \cos\theta_0$, $C = \cos y$, $D = \sin y$

Hence to compute S, θ, λ from x and y, first compute S from (24) and then θ and λ from (25) i.e.:

let $C = \cos y$, $D = \sin y$, $E = \sin x$, $A = C \sin\theta_0$, $B = D \cos\theta_0$.

Then

$$S = \arcsin(E/C) + \tfrac{1}{2}(S_1 + S_2)$$

$$\theta = \arcsin(A \cos S + B) \tag{26}$$

$$\lambda = \lambda_0 - \arcsin(C \sin S / \cos\theta)$$

DERIVATION OF THE EQUATIONS TO SPHERICAL HYPERBOLAS

Having established a rectangular spherical coordinate system on a great circle base line, we are now in a position to develop the equations of spherical hyperbolas referred to our rectangular system. Referring again to Figure 5, we restrict the point $Q(\theta, \lambda)$ or $Q(x,y)$ to the locus defined by demanding that the distances σ_1 and σ_2 from the points Q_2 and Q_1 respectively satisfy the condition

$$\begin{aligned} \sigma_1 - \sigma_2 &= 2c/e = 2a \\ 2c &= S_1 - S_2, \end{aligned} \tag{27}$$

where as before S_1, S_2 are the distances of Q_1, Q_2 respectively from $O(\theta_0, \lambda_0)$; e is a number such that $e > 1$.

From the spherical triangles MQQ_1, MQQ_2 one has

$$\begin{aligned} \cos\sigma_2 &= \cos r \cos c + \sin r \sin c \cos\alpha \\ \cos\sigma_1 &= \cos r \cos c - \sin r \sin c \cos\alpha \end{aligned} \tag{28}$$

Adding and substracting respective members of (28) obtain

$$\begin{aligned} \cos\sigma_1 + \cos\sigma_2 &= 2\cos r \cos c \\ \cos\sigma_1 - \cos\sigma_2 &= -2\sin r \sin c \cos\alpha \end{aligned} \tag{29}$$

By well known trigonometric identities and condition (27), equations (29) may be written

$$\cos\sigma_1 + \cos\sigma_2 = 2\cos\tfrac{1}{2}(\sigma_1+\sigma_2)\cos\tfrac{1}{2}(\sigma_1-\sigma_2) = 2\cos\tfrac{1}{2}(\sigma_1+\sigma_2)\cos a = 2(\cos r)(\cos c),$$

$$\cos\sigma_1 - \cos\sigma_2 = 2\sin\tfrac{1}{2}(\sigma_1+\sigma_2)\sin\tfrac{1}{2}(\sigma_1-\sigma_2) = 2\sin\tfrac{1}{2}(\sigma_1+\sigma_2)\sin a = -2(\sin r)(\sin c)\cos\alpha,$$

or

$$\begin{aligned} \cos\tfrac{1}{2}(\sigma_1+\sigma_2) &= \cos r \cos c/\cos a, \\ \sin\tfrac{1}{2}(\sigma_1+\sigma_2) &= \sin r \sin c \cos\alpha/\sin a. \end{aligned} \tag{30}$$

Squaring and adding respective members of (30), get

$$(\cos^2 r)(\cos^2 c/\cos^2 a) + (\sin^2 r \cos^2\alpha)(\sin^2 c/\sin^2 a) = 1. \tag{31}$$

Now in (31) place $\cos^2 r = 1/(1+\tan^2 r)$,

$\sin^2 r = \tan^2 r/(1+\tan^2 r)$, whence (31) may be written

$$\tan^2 r = \frac{\tan^2 a\,(\cos^2 a - \cos^2 c)}{\sin^2 c\,\cos^2\alpha - \sin^2 a} = \frac{\tan^2 a\,(\sin^2 c - \sin^2 a)}{\sin^2 c\,\cos^2\alpha - \sin^2 a} \tag{32}$$

Now (32) is the polar form of the equation to the spherical hyperbola.

From conversion formulas (15) we have

$$\begin{aligned} \tan^2 r &= (\sin^2 x + \sin^2 y)/(1 - \sin^2 x - \sin^2 y), \\ \cos^2\alpha &= \sin^2 x/(\sin^2 x + \sin^2 y) \end{aligned} \tag{33}$$

and substitutions for $\tan^2 r$, $\cos^2\alpha$ from (33) in (32) give the rectangular equation to the spherical hyperbola

$$\sin^2 x = \frac{\sin^2 a \cos^2 c}{\sin^2 c - \sin^2 a} \cdot \sin^2 y + \sin^2 a. \tag{34}$$

THE POLAR EQUATION OF SPHERICAL HYPERBOLAS WITH ORIGIN AT A FOCUS

If we choose the given point Q_1 (θ_1, λ_1) of the great circle base line as origin of co-ordinates and a focus, then the following figure may be abstracted from Figure 5:

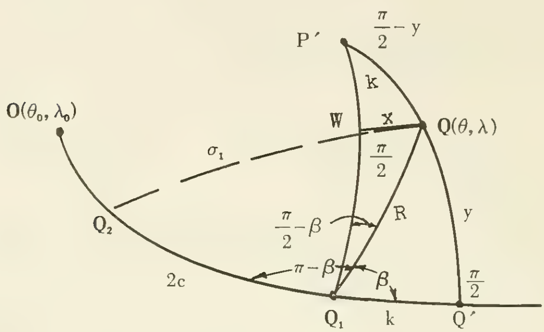


Figure 6.

The polar radius is now $R = \sigma_2$, β is the angle between R and Q_1Q'. $k = Q_1Q' = S - S_1$. From spherical triangle Q_2QQ_1 we find $\cos\sigma_1 = \cos R \cos 2c - \sin R \sin 2c \cos\beta$, (35)

and from (27) $\sigma_1 - R = 2a$, whence

$$\cos(\sigma_1 - R) = \cos\sigma_1 \cos R + \sin\sigma_1 \sin R = \cos 2a, \tag{36}$$

$$\sin(\sigma_1 - R) = \cos\sigma_1 \sin R + \sin\sigma_1 \cos R = \sin 2a.$$

Multiply the first of (36) by sin R, the second by cos R and add respective members to solve for

$$\sin\sigma_1 = \cos 2a \sin R + \sin 2a \cos R. \tag{37}$$

Square and add respective members of (35) and (37) to get

$$(\cos R \cos 2c - \sin R \sin 2c \cos\beta)^2 + (\cos 2a \sin R + \sin 2a \cos R)^2 = 1. \tag{38}$$

Multiply every term of (38) by $\sec^2 R$, whence it may be written

$$(\cos 2c - \tan R \sin 2c \cos\beta)^2 + (\cos 2a \tan R + \sin 2a)^2 = \sec^2 R = 1 + \tan^2 R. \tag{39}$$

Expanding (39) and writing as a quadratic in tan R find

$$\tan^2 R\,(\sin^2 2c \cos^2\beta - \sin^2 2a) + 2\tan R\,(\sin 2a \cos 2a - \sin 2c \cos 2c \cos\beta) \tag{40}$$
$$+ \cos^2 2c - \cos^2 2a = 0.$$

Now equation (40) factors into $[\tan R (\sin 2c \cos \beta + \sin 2a) - (\cos 2c + \cos 2a)]$.

$$[\tan R (\sin 2c \cos \beta - \sin 2a) - (\cos 2c - \cos 2a)] = 0. \tag{41}$$

Whence

$$\tan R = \frac{\cos 2c + \cos 2a}{\sin 2c \cos \beta + \sin 2a}, \quad \tan R = \frac{\cos 2c - \cos 2a}{\sin 2c \cos \beta - \sin 2a}$$

or

$$\tan R = \frac{\cos 2c \pm \cos 2a}{\sin 2c \cos \beta \pm \sin 2a}, \tag{42}$$

where either the (two plus signs) or (two minus) signs are taken together.

Equation (42) is the polar equation to spherical hyperbolas referred to a focus as pole. We now derive expressions for the spherical rectangular coordinates x, y as functions of the polar coordinates R, β.

From right triangles WP'Q, WQQ_1, Q_1QQ' (Figure 6) find

$$\sin x = \sin R \cos \beta,$$

$$\sin y = \sin R \sin \beta. \tag{43}$$

$$\sin x = \sin k \cos y;$$

$$\cos R = \cos k \cos y. \tag{44}$$

Equations (43) are similar to equations (14) and provide the conversions from polar to rectangular coordinates, i.e. from (43)

$$\sin R = (\sin^2 x + \sin^2 y)^{1/2}, \tag{45}$$

$$\tan \beta = \sin y / \sin x.$$

Since moving the origin from M to Q_1 (see Figure 5) is only a translation along the x-axis, there is no change in y, but x is changed. Hence from (44) and the relations (23) and (26) we can write when the origin is at Q_1, $k = S - S_1$:

FORMULAS FOR COMPUTATION OF S, x, y, FROM θ AND λ

$$\sin y = \cos \theta_0 \sin \theta - \sin \theta_0 \cos \theta \cos (\lambda_0 - \lambda)$$

$$\tan S = \frac{\sin \theta_0 \cos \theta \sin (\lambda_0 - \lambda)}{\sin \theta - \cos \theta_0 \sin y} = \frac{\cos \theta \sin (\lambda_0 - \lambda)}{\cos f} \tag{46}$$

$$= \frac{\cos \theta \sin (\lambda_0 - \lambda)}{\sin \theta_0 \sin \theta + \cos \theta_0 \cos \theta \cos (\lambda_0 - \lambda)}$$

$$\sin x = \sin k \cos y = \sin (S - S_1) \cos y$$

FORMULAS FOR COMPUTATION OF S, θ, λ FROM x AND y

Let $C = \cos y$, $D = \sin y$, $E = \sin x$, $A = C \sin \theta_0$, $B = D \cos \theta_0$, then

$$S = \arcsin (E/C) + S_1$$

$$\theta = \text{arc sin } (A \cos S + B) \tag{47}$$

$$\lambda = \lambda_0 - \text{arc sin } (C \sin S/\cos \theta)$$

AN ALTERNATIVE EQUATION TO THE SPHERICAL HYPERBOLA WITH ORIGIN AT A FOCUS

If $S = \frac{1}{2}(a_0 + b_0 + c_0)$ in the spherical triangle

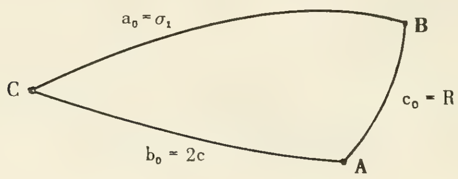


Figure 7.

$$\text{then } \tan^2 \tfrac{1}{2}A = \frac{\sin (s - b_0) \sin (s - c_0)}{\sin S \sin (s - a_0)}, [6]. \tag{48}$$

Referring to figure 6, $a_0 = \sigma_1, b_0 = 2c, c_0 = R$: and from (27) we have the conditions

$\sigma_1 - R = 2a, \sigma_1 + R = 2(R + a).$

Hence

$$s = \tfrac{1}{2}(\sigma_1 + R) + c = R + a + c,$$

$$s - a_0 = \tfrac{1}{2}(R - \sigma_1) + c = c - a, \tag{49}$$

$$s - b_0 = R + a - c, \quad S - c_0 = c + a$$

$$A = \pi - \beta, \tan \tfrac{1}{2}A = \tan (\pi/2 - \beta/2) = \cot \beta/2$$

With the values from (49) placed in (48) find

$$\tan^2 \beta/2 = \frac{\sin(c - a) \sin (R + c + a)}{\sin (c + a) \sin (R - c + a)}, \tag{50}$$

which is the desired alternative form, [7].

CORRESPONDING PLANE HYPERBOLA EQUIVALENTS

For the plane case and analogous reference system, Figure 5 becomes

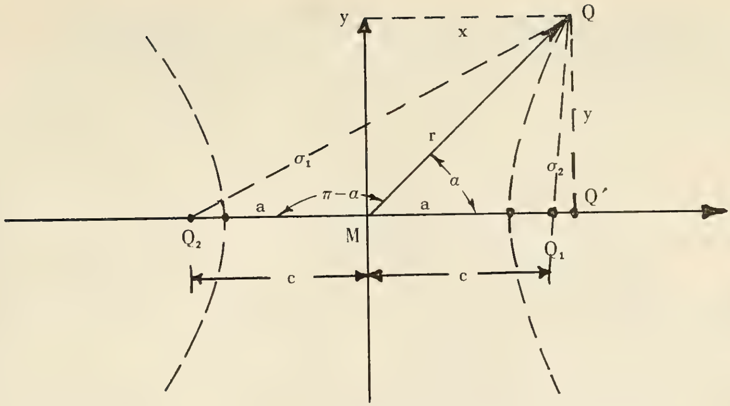


Figure 8.

Given the condition $\sigma_1 - \sigma_2 = 2a$

By the law of cosines applied to triangles MQQ_1 MQQ_2

$$\sigma_2^2 = r^2 + c^2 - 2rc \cos \alpha,\ \sigma_1^2 = r^2 + c^2 + 2rc \cos \alpha$$

whence $\sigma_1^2 + \sigma_2^2 = 2(r^2 + c^2),\ \ \sigma_1^2\ \sigma_2^2 = (r^2 + c^2)^2 - 4r^2\ c^2 \cos^2\alpha$ (51)

Now by squaring both sides of $\sigma_1 - \sigma_2 = 2a$ obtain

$$\sigma_1^2 - 2\sigma_1\,\sigma_2 + \sigma_2^2 = 4a^2 \text{ whence}$$

$$(\sigma_1^2 + \sigma_2^2 - 4a^2)^2 = 4\sigma_1^2\sigma_2^2 \tag{52}$$

With the values of $\sigma_1^2 + \sigma_2^2$, $\sigma_1^2\sigma_2^2$ from (51) placed in (52) obtain

$$[2(r^2 + c^2) - 4a^2]^2 = 4[(r^2 + c^2)^2 - 4r^2c^2 \cos^2 \alpha]. \tag{53}$$

Expanding (53) find

$$r^2\ c^2 \cos^2 \alpha - a^2\ r^2 - a^2\ c^2 + a^4 = 0$$

or
$$r^2 = \frac{a^2(c^2 - a^2)}{c^2 \cos^2 \alpha - a^2} \tag{54}$$

To transform to rectangular equation we have $x = r \cos \alpha$, $y = r \sin \alpha$, or $r^2 = x^2 + y^2$, $\tan \alpha = \frac{y}{x}$, $\cos^2 \alpha = x^2/(x^2 + y^2)$ and these values of r^2 and $\cos^2\alpha$ placed in (54) give

$$x^2 = \frac{a^2\ y^2}{c^2 - a^2} + a^2 \tag{55}$$

as corresponding rectangular equation.

If the focus Q_1 is to be the origin and $\sigma_2 = R$, the radius for polar coordinates, and β the angle which R makes with the positive x-axis, i.e. β is the angle QQ_1Q', then our plane figure is as follows:

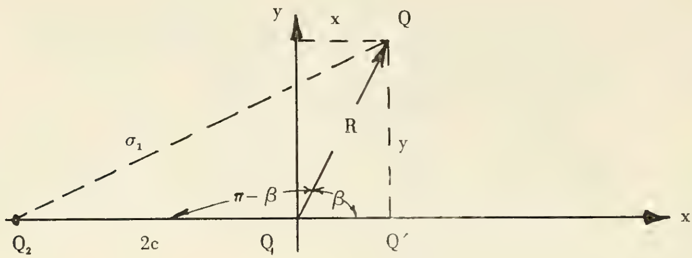


Figure 9.

By the law of cosines in triangle Q_2QQ_1

$$\sigma_1^2 = 4c^2 + R^2 + 4cR\cos\beta \tag{56}$$

From the condition $\sigma_1 - R = 2a$, $\sigma_1 = R + 2a$, and this value of σ_1 placed in (56) gives $(R + 2a)^2 = 4c^2 + R^2 + 4cR\cos\beta$, which when expanded gives

$$R = \frac{a^2 - c^2}{c\cos\beta - a} \tag{57}$$

For the alternative form of (57), we have the well known formula

$$\tan^2 \tfrac{1}{2}A = \frac{(s - b_0)(s - c_0)}{s(s - a_0)}, \text{ where } 2s = a_0 + b_0 + c_0 \tag{58}$$

Here $a_0 = \sigma_1$, $b_0 = R$, $c_0 = 2c$, $A = \pi - \beta$,

Hence: $s = a + c + R$, $s - a_0 = c - a$, $s - b_0 = a + c$, $s - c_0 = a - c + R$,

whence $$\tan^2 \tfrac{1}{2}\beta = \frac{(c - a)(R + c + a)}{(c + a)(R - c + a)}, \tag{59}$$

which is an alternative form of (57).

Now (54), (55), (57) and (59) could have been obtained directly from (32), (34),(42) and (50) by replacing correctly the trigonometric functions of lengths by corresponding lengths, i.e. $\tan a = \sin a = a$, $\cos a = 1$, etc. We place them side by side for direct comparison in the following table which will also serve as a summary for both:

SPHERICAL HYPERBOLA FORMULAS AND PLANE EQUIVALENTS, [7]

SPHERICAL	PLANE (60)
(1) $\tan^2 r = \dfrac{\tan^2 a\,(\sin^2 c - \sin^2 a)}{\sin^2 c \cos^2 \alpha - \sin^2 a}$	$r^2 = \dfrac{a^2(c^2 - a^2)}{c^2\cos^2\alpha - a^2}$
(2) $\sin^2 x = \dfrac{\sin^2 a \cos^2 c}{\sin^2 c - \sin^2 a} \sin^2 y + \sin^2 a$	$x^2 = \dfrac{a^2 y^2}{c^2 - a^2} + a^2$
(3) $\tan R = \dfrac{\cos 2c \pm \cos 2a}{\sin^2 c \cos\beta \pm \sin 2a}$	$R = \dfrac{a^2 - c^2}{c\cos\beta - a}$
(4) $\tan^2(\beta/2) = \dfrac{\sin(c-a)\sin(R+c+a)}{\sin(c+a)\sin(R-c+a)}$	$\tan^2(\beta/2) = \dfrac{(c-a)(R+c+a)}{(c+a)(R-c+a)}$

In (1) and (2) of equations (60), the origin of coordinates is the midpoint M_1, of the segment Q_1Q_2, see Figure 5. (3) and (4) are two polar forms with origin at a Focus Q_1, see Figures (5) and (6).

REFERENCES

[6] Chauvenet, Plane and Spherical Trigonometry, 1871, page 158.

[7] Equations (32), (34), (42), (50) to spherical hyperbolas are essentially those given without derivation in LORAN, Pierce, McKenzie, Woodward, McGraw Hill 1948, pages 173, 175.

DEVELOPMENT: DISTANCE FORMULAE;

SECTION 3. DISTANCE COMPUTATIONS AND CONVERSIONS; AZIMUTHS

If we are given two points $P_1(\phi_1, \lambda_1)$, $P_2(\phi_2, \lambda_2)$ on the ellipsoid of reference as shown in Figure 10, we may compute distances and azimuths according to known or given elements. That is we may compute the geographic coordinates of the point $P_2(\phi_2, \lambda_2)$ if we know the geographic coordinates of P_1 (ϕ_1, λ_1) the distance between P_1 and P_2, and the azimuth from P_1 to P_2. This is the direct problem and the one most important in Geodesy relative to establishing triangulation control nets. If the coordinates of both P_1 and P_2 are given, the distance between them and the azimuths can be computed. This is the inverse problem, and the one concerned primarily in electronic positioning systems as Loran.

Since there are several possible curves connecting the points P_1 and P_2 on the ellipsoid along which distances would differ very little, for instance – the geodesic, the normal sections, the great elliptic arc, the curve of alinement, etc. – criteria for selection would be simplicity in computations relative to required accuracy. Also to be considered are other useful geometric quantities associated with the configuration and expressible in terms of common computational parameters. (See Figure 11).

The shortest distance is always the geodesic or the geodetic line between P_1 and P_2. It is usually a space curve (that is it has a first and second curvature at each point). For instance on the reference ellipsoid, the equator and the meridians are the only plane geodesics, [8].

Now in Figure 10, the point $P_0(\phi_0, \lambda_0)$ is the vertex of the great elliptic arc, that is P_0 is the point where the great elliptic arc is orthogonal to a meridian. The goedesic, or geodetic line, between P_1 and P_2 also has a vertex where it is orthogonal to a meridian. Since the geodesic is a space curve and climbs nearer to the ellipsoid pole, T_0, than any of the other representative curves (if P_1 and P_2 were ends of a diameter of the equator, the geodesic would be the elliptic meridian through P_1 and P_2 since it is shorter than the equator), the vertex of the geodesic is closer to T_0 than is P_0. Unfortunately the geographic coordinates of the geodesic vertex cannot be expressed simply in terms of the geographic coordinates of P_1 and P_2, hence an approximation scheme, usually iterative, is used. [9] The computations are usually quite lengthy for long lines. Many schemes and formulae have been devised to approximate the geodesic and studies have been made comparing them. [21] The geodetic line is of most interest to the geodesist proper, since he is primarily concerned with closure on a particular ellipsoid of reference of large arcs and areas of triangulation, hence the geodesic or geodetic line and geodetic azimuths on the ellipsoid are consonant with his mathematical model.

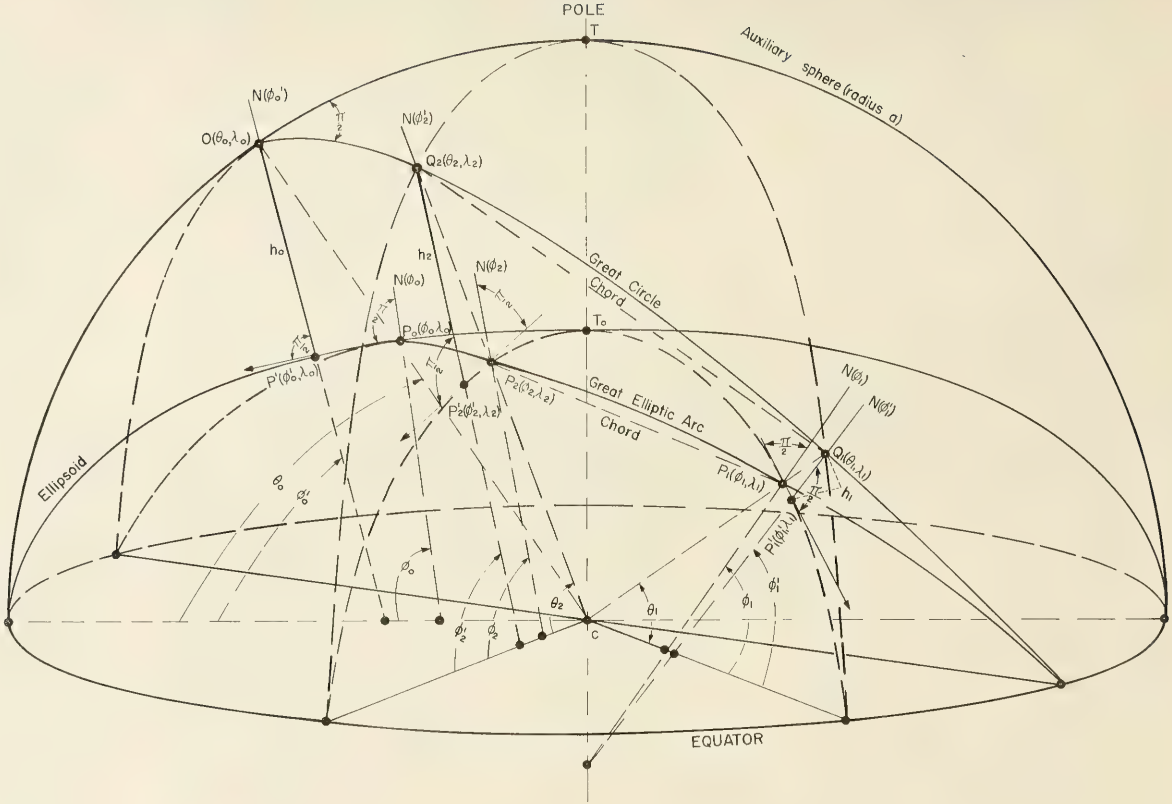


Figure 10. Corresponding distances on the reference ellipsoid and the auxiliary sphere.

OPERATIONAL APPLICATIONS

Requirements, accuracy wise, with respect to geodetic data obviously depend on the particular guidance system employing it. If some guidance, particularly external, is to be provided a missile, its initial launch requirements are not as critical as say for a purely ballistic missile. Since it has yet to be demonstrated that the flight of missiles are geodesic or that the traces of the trajectories upon the ellipsoid of reference are geodesics, distances can be computed by any method which will give results within the capability of the particular system. Since alinement is usually with respect to a local vertical and a "bearing", the normal section azimuth, the angle of depression of the chord below the horizon and the maximum separation between the chord and the surface are all useful associated quantities which can be "integrated" in the computations for distance as will subsequently be shown in the discussion of distance computations along the great elliptic arc. This configuration is shown in Figure 11 as abstracted from Figure 10.

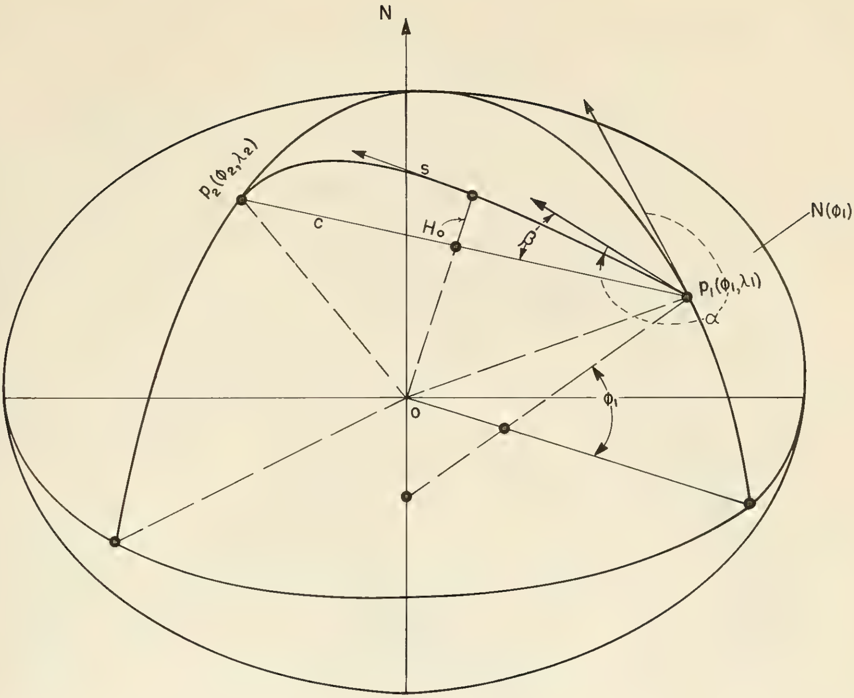
HYPERBOLIC MEASURING SYSTEMS

For Loran systems, the earth must be considered an oblate ellipsoid or spheroid, but the nearest hundred feet is probably close enough particularly on long lines. [7], page 170. Hence a computational system is desirable which provides modifications to spherical elements, i.e. functions of spherical arc lengths so that the auxiliary sphere of the particular spheroid of reference can be used since the hyperbolic propagation of systems as Loran may be worldwide as base lines are added or extended. Also to be considered is the use of such computational systems in local areas as for oceanographic surveying and corresponding adaptation to a local sphere of reference. Azimuth computations should be independent, except for dependence on spherical arc length, so that one can have readily the Normal plane section azimuths as well as geodetic azimuths. Finally the system should be easily adapted to local area work in terms of plane coordinates. This can probably best be accomplished through the series of projections, all conformal; spheroid to aposphere, aposphere to sphere, sphere to plane. [8].

The present investigation will center about the configuration depicted in Figure 12 which shows the relationships, exaggerated; between the Normal sections, The Great Elliptic Section, The Geodesic, and the Chord between two points Q_1, Q_2 on the ellipsoid. We begin by deriving the formulae for the Normal Section Azimuths and the Great Elliptic Arc Azimuths.

NORMAL SECTION AZIMUTHS

The normal section azimuths are shown in Figure 13, as extended from Figure 11. The spheroid has been referred to its center as origin of rectangular coordinates, with the reference plane – xz containing the point $Q_1(\phi_1, \lambda_1)$ as shown. The z-axis is the polar axis of the spheroid



α = Normal Section Azimuth at P_1 (from North)
S = Arc length - Geodetic distance
C = Chord length, $P_1 P_2$
β = Angle of depression of C below horizon at P_1
H_o = Maximum separation of arc S and chord C

Figure 11. Relationship between arc length, normal section azimuth, chord length, angle of depression of the chord below the horizon, maximum separation of arc and chord.

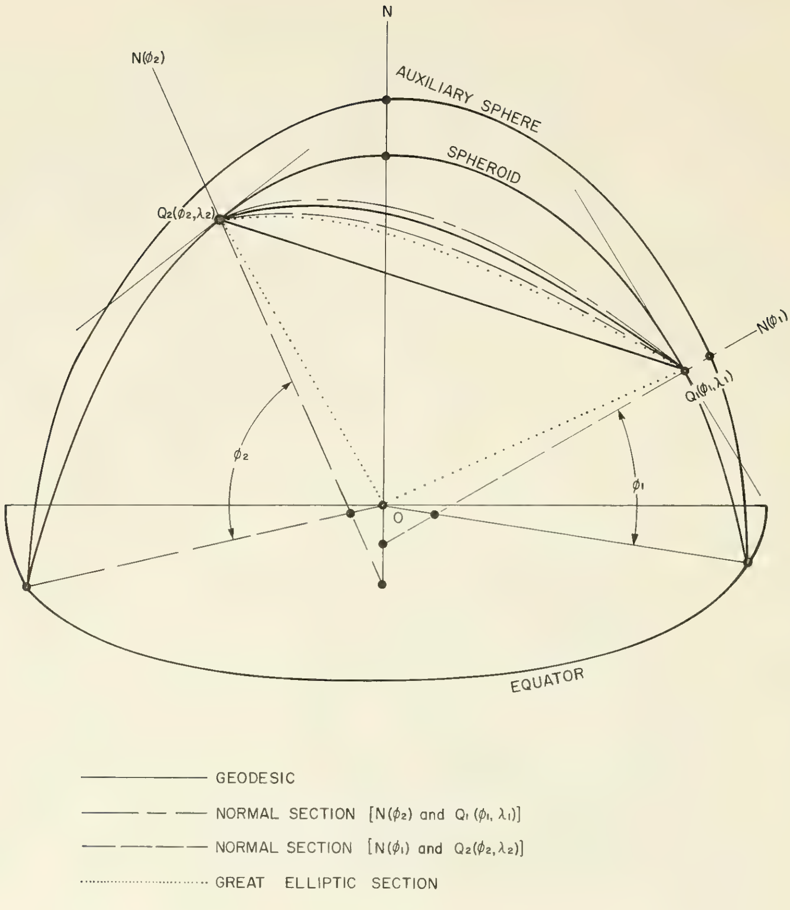


Figure 12. Relationships relative to the pole on the ellipsoid of reference, of the geodesic, normal sections, and great elliptic section.

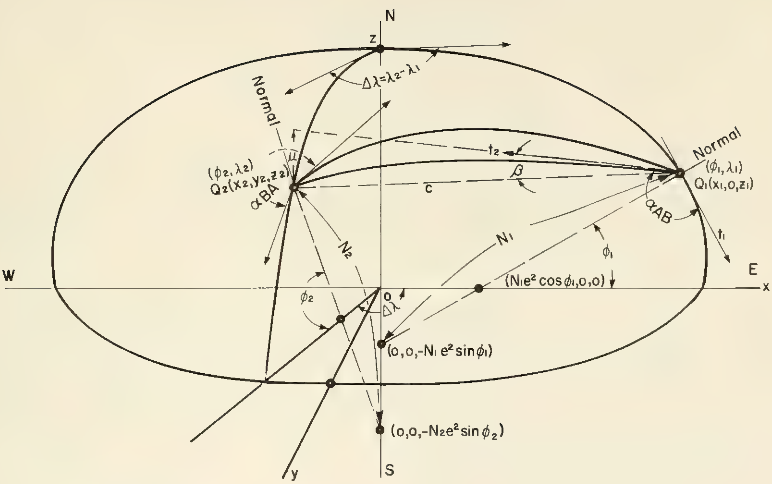


Figure 13. The normal section azimuths.

and the y-axis is then in the plane of the equator — the xy-plane is the equatorial plane of the ellipsoid. In this coordinate system the points $Q_1(\phi_1, \lambda_1)$, $Q_2(\phi_2, \lambda_2)$ have the rectangular coordinates:

$$\begin{aligned} Q_1: x_1 &= N_1 \cos\phi_1 & \qquad Q_2: x_2 &= N_2 \cos\phi_2 \cos\Delta\lambda \\ y_1 &= 0 & y_2 &= N_2 \cos\phi_2 \sin\Delta\lambda \\ z_1 &= N_1(1-e^2)\sin\phi_1 & z_2 &= N_2(1-e^2)\sin\phi_2 \end{aligned} \tag{1}$$

The rectangular equation to the ellipsoid is

$$(1-e^2)(x^2+y^2)+z^2-a^2(1-e^2)=0, \tag{2}$$

where a, e are respectively the semimajor axis and eccentricity of the meridian ellipse.

The tangent plane to (2) at any point (x_1, y_1, z_1) is

$$(1-e^2)(xx_1+yy_1)+zz_1-a^2(1-e^2)=0. \tag{3}$$

Hence the tangent plane at Q_1 is, from (1) and (3)

$$xN_1\cos\phi_1 + z N_1 \sin\phi_1 - a^2 = 0. \tag{4}$$

The equation of the plane containing the normal at Q_1 and the point Q_2 is determined by Q_2 and the points $(N_1 e^2 \cos\phi_1, 0, 0)$, $(0, 0, -N_1e^2\sin\phi_1)$, see Figure 13. With the coordinates of Q_2 from (1) we can write the equation as

$$\begin{vmatrix} x & y & z & 1 \\ N_2\cos\phi_2\cos\Delta\lambda & N_2\cos\phi_2\sin\Delta\lambda & N_2(1-e^2)\sin\phi_2 & 1 \\ N_1e^2\cos\phi_1 & 0 & 0 & 1 \\ 0 & 0 & -N_1e^2\sin\phi_1 & 1 \end{vmatrix} = 0,$$

which upon expansion may be written

$$Ax + By - Cz - D = 0$$

where $A = N_2 \sin\phi_1 \cos\phi_2 \sin\Delta\lambda$ (5)

$$B = (N_1\sin\phi_1 - N_2\sin\phi_2)e^2\cos\phi_1 + N_2(\sin\phi_2\cos\phi_1 - \sin\phi_1\cos\phi_2\cos\Delta\lambda)$$

$$C = N_2\cos\phi_1\cos\phi_2\sin\Delta\lambda$$

$$D = N_1N_2e^2\sin\phi_1\cos\phi_1\cos\phi_2\sin\Delta\lambda.$$

Now the direction cosines p, q, r of the intersection of two planes $A_1x + B_1y + C_1z = D_1$, $A_2x + B_2y + C_2z = D_2$ are given by

$$p = (B_1C_2 - B_2C_1)/d, \quad q = (C_1A_2 - A_1C_2)/d, \quad r = (A_1B_2 - A_2B_1)/d \tag{6}$$

where $d = [(B_1C_2 - B_1C_2)^2 + (C_1A_2 - A_1C_2)^2 + (A_1B_2 - A_2B_1)^2]^{1/2}$.

Note from figure 13 that the tangent, t_1, to the meridian at Q_1 lies in the plane y = 0 and that defined by equation (4). To apply (6) to these two planes we have respectively $A_1 = C_1 = D_1 = 0$, $B_1 = 1$; $A_2 = N_1\cos\phi_1$, $B_2 = 0$, $C_2 = N_1\sin\phi_1$, $D_2 = a^2$ and (6) gives the direction cosines of t_1 as $p_1 = \sin\phi_1$, $q_1 = 0$, $r_2 = -\cos\phi_1$. (7)

(These were apparent from inspection of Figure 13 but illustrate the use of (6)).

From Figure 13, the tangent t_2 to the elliptic section lying in the plane (5) is the line of intersection of the planes (4) and (5). From (4) and (5) we have respectively $A_1 = N_1 \cos \phi_1$, $B_1 = \theta$, $C_1 = N_1 \sin \phi_1$; $A_2 = A$, $B_2 = B$, $C_2 = -C$ and applying (6) find the direction cosines of t_2 to be

$$P_2 = (-B \sin \phi_1)/d, \; q_2 = (A \sin \phi_1 + C \cos \phi_1)/d, \; r_2 = (B \cos \phi_1)/d$$

where $d = [B^2 + (A \sin \phi_1 + C \cos \phi_1)^2]^{1/2}$. (8)

The forward azimuth α_{AB} from Q_1 to Q_2, as shown in Figure 13, is the angle reckoned clockwise from south between the tangents t_1 and t_2. Hence from (7) and (8)

$$\cos \alpha_{AB} = p_1p_2 + q_1q_2 + r_1r_2 = -\frac{B}{d}\sin^2\phi_1 - \frac{B}{d}\cos^2\phi_1 = -\frac{B}{d}, \tag{9}$$

$$d = [B^2 + (A \sin \phi_1 + C \cos \phi_1)^2]^{1/2}$$

Since $\cot \alpha_{AB} = \cos \alpha_{AB}/(1 - \cos^2\alpha_{AB})^{1/2}$ we have from (9) that

$$\cot \alpha_{AB} = -B/(d^2 - B^2)^{1/2}, \tag{10}$$

Now $d^2 - B^2 = B^2 + (A \sin \phi_1 + C \cos \phi_1)^2 - B^2 = (A \sin \phi_1 + C \cos \phi_1)^2$, so $\sqrt{d^2 - B^2} = A \sin \phi_1 + C \cos \phi_1$ and (10) may be written

$$\cot \alpha_{AB} = -B/(A \sin \phi_1 + C \cos \phi_1). \tag{11}$$

With the values of A, B, C from (5), equation (11) may be written as

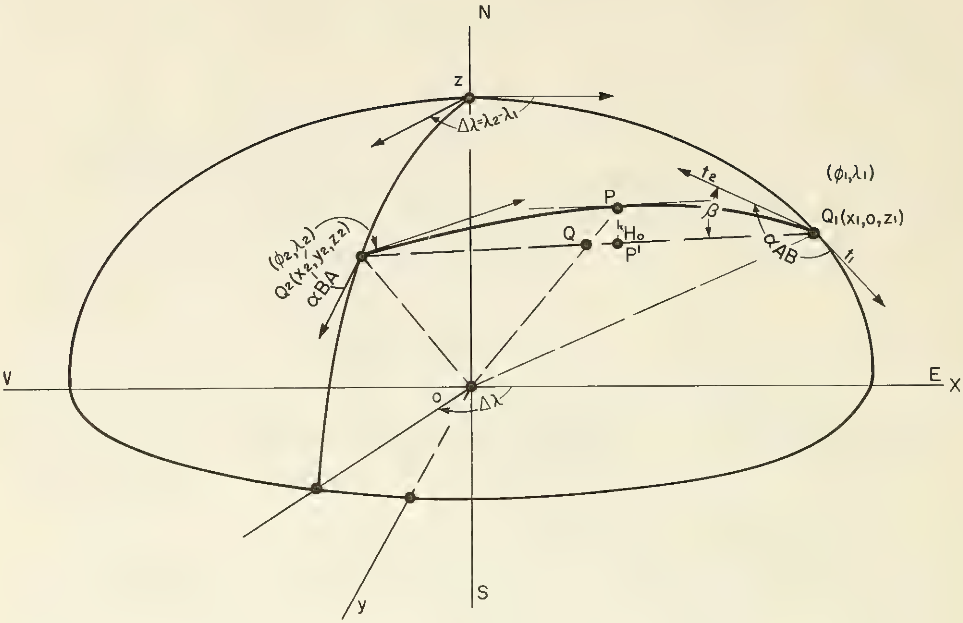
$$\cot \alpha_{AB} = \frac{[\sin \phi_2 - (N_1/N_2) \sin_1\phi\,]\,e^2 \cos \phi_1 \sec \phi_2 + (\sin \phi_1 \cos \Delta\lambda - \tan \phi_2 \cos \phi_1).}{\sin \Delta\lambda} \tag{12}$$

Referring again to figure 13, it is seen that from considerations of symmetry, we have only to interchange the subscripts 1 and 2 and change $\Delta\lambda$ to $-\Delta\lambda$ in (12) to obtain $\cot_{BA}$ (the back azimuth on the other normal section). We thus obtain from (12)

$$\cot \alpha_{BA} = -\frac{[\sin \phi_1 - (N_2/N_1) \sin \phi_2]e^2 \cos \phi_2 \sec \phi_1 + (\sin \phi_2 \cos \Delta\lambda - \tan \phi_1 \cos \phi_2)}{\sin \Delta\lambda} \tag{13}$$

GREAT ELLIPTIC SECTION AZIMUTHS

Figure 14 shows the great elliptic section and azimuths as abstracted from Figure 12. The same coordinate system is used as in Figure 13 so that most of the equations developed with the normal section azimuths can be used. The angle α_{AB} between the tangents t_1 and t_2 is the forward azimuth required. We already have the direction cosines of t_1 see equations (7). The tangent t_2 is the intersection of the great elliptic plane with the tangent plane at Q_1, equation (4). The equation of the great elliptic plane through Q_1, Q_2, using equations (1), is given by the determinant



GREAT ELLIPTIC SECTION AZIMUTHS
AND ASSOCIATED GEOMETRY

P-point of maximum separation, chord and arc

H_o- maximum separation of chord and arc

Figure 14. The great elliptic section azimuths.

$$\begin{vmatrix} x & y & z & 1 \\ N_1\cos\phi_1 & 0 & N_1(1-e^2)\sin\phi_1 & 1 \\ N_2\cos\phi_2\cos\Delta\lambda & N_2\cos\phi_2\sin\Delta\lambda & N_2(1-e^2)\sin\phi_2 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 0 ,$$

which when expanded reduces to

$$\begin{aligned} &Ax + By - Cz = 0, \\ &A = (1-e^2)\tan\phi_1\sin\Delta\lambda \\ &B = (1-e^2)(\tan\phi_2 - \tan\phi_1\cos\Delta\lambda) \qquad (\Delta\lambda = \lambda_2 - \lambda_1) \\ &C = \sin\Delta\lambda \end{aligned} \tag{14}$$

Since equation (11) was developed for generalized coefficients A, B, C we have only to substitute the values of A, B, C from (14) in (11) to obtain after some algebraic manipulation,

$$\cot\alpha_{AB} = (1-e^2)\,\frac{N_1^2}{a^2}\,\frac{(\tan\phi_1\cos\Delta\lambda - \tan\phi_2)\cos\phi_1}{\sin\Delta\lambda} \tag{15}$$

By symmetrical interchange of subscripts and replacing $\Delta\lambda$ by $-\Delta\lambda$, we obtain $\cot\alpha_{BA}$ from (15) as

$$\cot\alpha_{BA} = (1-e^2)\,\frac{N_2^2}{a^2}\,\frac{(\tan\phi_1 - \tan\phi_2\cos\Delta\lambda)\cos\phi_2}{\sin\Delta\lambda} \tag{16}$$

Equations (15) and (16) represent the azimuths of the great elliptic section as shown in Figure 14.

NORMAL SECTION AND GREAT ELLIPTIC SECTION AZIMUTHS IN TERMS OF PARAMETRIC LATITUDE θ

From the transformation equations $\tan\theta = (1-e^2)^{1/2}\tan\phi$, $\cos\theta = \frac{N}{a}\cos\phi$, $\sin\theta = \frac{(1-e^2)^{1/2}}{a} N\sin\phi$, $(1-e^2\cos^2\theta)^{1/2} = \frac{(1-e^2)^{1/2}}{a} N$

applied to equations (12), (13), (15), 16) we have the normal section and great elliptic section azimuths in terms of parametric latitude.

Normal Section Azimuths in terms of θ.

$$\begin{aligned} \cot\alpha_{AB} &= +\frac{\sin\theta_1\cos\Delta\lambda - \cos\theta_1\tan\theta_2 + e^2(\sin\theta_2 - \sin\theta_1)\cos\theta_1\sec\theta_2}{(1-e^2\cos^2\theta_1)^{1/2}\sin\Delta\lambda} \\ \cot\alpha_{BA} &= -\frac{\sin\theta_2\cos\Delta\lambda - \cos\theta_2\tan\theta_1 + e^2(\sin\theta_1 - \sin\theta_2)\cos\theta_2\sec\theta_1}{(1-e^2\cos^2\theta_2)^{1/2}\sin\Delta\lambda} \end{aligned} \tag{17}$$

Great Elliptic Section Azimuths in terms of θ

$$\cot \alpha_{AB} = + \frac{(\tan \theta_1 \cos \Delta\lambda - \tan \theta_2)(\cos \theta_1)(1 - e^2 \cos^2 \theta_1)^{1/2}}{\sin \Delta\lambda}$$

$$\cot \alpha_{BA} = + \frac{(\tan \theta_1 - \tan \theta_2 \cos \Delta\lambda)(\cos \theta_2)(1 - e^2 \cos^2 \theta_2)^{1/2}}{\sin \Delta\lambda} \tag{18}$$

GREAT ELLIPTIC ARC DISTANCE

Referring to Figure 9, it is seen that the great elliptic arc is orthogonal to a meridian at a point $P_0(\phi_0, \lambda_0)$ which is the vertex of the great elliptic arc determined by the points $P_1(\phi_1, \lambda_1)$, $P_2(\phi_2, \lambda_2)$ on the ellipsoid. The equation of the great elliptic plane through P_1 and P_2 is given by equations (14). Now a meridional plane orthogonal to (14) has an equation of the form $Bx - Ay = 0$ and the rectangular coordinates of $P_0(\phi_0, \lambda_0)$ must satisfy both planes. From (1), the rectangular coordinates of $P_0(\phi_0, \lambda_0)$ are $x_0 = N_0 \cos \phi_0 \cos \Delta\lambda_0$, $y_0 = N_0 \cos \phi_0 \sin \Delta\lambda_0$, $z = N_0(1 - e^2) \sin \phi_0$ and these placed in $Bx - Ay = 0$ and (14) give

$$\begin{aligned} &B \cos \Delta\lambda_0 - A \sin \Delta\lambda_0 = 0, \\ &A \cos \Delta\lambda_0 + B \sin \Delta\lambda_0 = C(1 - e^2) \tan \phi_0 . \end{aligned} \tag{19}$$

From the first of (19) find $\tan \Delta\lambda_0 = B/A$, whence $\sin \Delta\lambda_0 = B/(A^2 + B^2)^{1/2}$ and these values placed in the second of (19) give $\tan \phi_0 = (A^2 + B^2)^{1/2}/C(1 - e^2)$,

$$\sin \phi_0 = \tan \phi_0 / (1 + \tan^2 \phi_0)^{1/2} = \left(\frac{A^2 + B^2}{A^2 + B^2 + C^2(1 - e^2)^2} \right)^{1/2}, \tag{20}$$

$\tan \Delta\lambda_0 = B/A$.

With the values of A, B, C from (14), equations (20) may be written

$$\sin \phi_0 = \left(\frac{\tan^2 \phi_1 - 2 \tan \phi_1 \tan \phi_2 \cos \Delta\lambda + \tan^2 \phi_2}{\tan^2 \phi_1 - 2 \tan \phi_1 \tan \phi_2 \cos \Delta\lambda + \tan^2 \phi_2 + \sin^2 \Delta\lambda} \right)^{1/2}, \tag{21}$$

$$\tan \Delta\lambda_0 = (\cot \phi_1 \tan \phi_2 - \cos \Delta\lambda)/\sin \Delta\lambda,$$

$$\tan \phi_0 = (\tan^2 \phi_1 + \tan^2 \phi_2 - 2 \tan \phi_1 \tan \phi_2 \cos \Delta\lambda)^{1/2}/\sin \Delta\lambda .$$

From the second of equations (19), dropping the subscript zero and differentiating we obtain

$$(-A \sin \Delta\lambda + B \cos \Delta\lambda)(d\,\Delta\lambda) = C(1 - e^2) \sec^2 \phi \, d\phi. \tag{22}$$

By solving $A \cos \Delta\lambda + B \sin \Delta\lambda = C(1 - e^2) \tan \phi$ with the identity $\sin^2 \Delta\lambda + \cos^2 \Delta\lambda = 1$, find

$$\sin \Delta\lambda = - \frac{BC(1 - e^2) \tan \phi + A[(A^2 + B^2) - C^2(1 - e^2)^2 \tan^2 \phi]^{1/2}}{A^2 + B^2}, \tag{23}$$

$$\cos \Delta\lambda = \frac{-AC(1 - e^2) \tan \phi + B[(A^2 - B^2) - C^2(1 - e^2)^2 \tan^2 \phi]^{1/2}}{A^2 + B^2} .$$

From (23) one has then

$- A \sin \Delta\lambda + B \cos \Delta\lambda = [(A^2 + B^2) - C^2(1 - e^2)^2 \tan^2\phi]^{1/2}$ and this value placed in (22) gives

$$(d\,\Delta\lambda) = \frac{C(1 - e^2) \sec^2 \phi\, d\,\phi}{[(A^2 + B^2) - C^2(1 - e^2)^2 \tan^2\phi]^{1/2}} \tag{24}$$

whence, by means of relations (20) and trigonometric identities,

$$(d\Delta\lambda)^2 = \frac{C^2(1 - e^2)^2 \sec^4\phi d\phi^2}{A^2 + B^2 - C^2(1 - e^2)^2 \tan^2\phi} = \frac{\sec^4\phi\, d\,\phi^2}{\dfrac{A^2 + B^2}{C^2(1 - e^2)^2} - \tan^2\phi}$$

$$= \frac{\sec^4\phi\, d\phi^2}{\tan^2\phi_0 - \tan^2\phi} = \frac{\sec^4\phi\, d\,\phi^2}{\sec^2\phi_0 - \sec^2\phi}\ . \tag{25}$$

Now the linear element of the spheroid is, [8] page 62,

$$ds^2 = \left[\sec^2\phi\, d\,\phi^2 + \left(\frac{N}{R}\right)^2 (d\,\Delta\lambda)^2\right] R^2 \cos^2\phi, \tag{26}$$

where $R = a(1 - e^2)/(1 - e^2 \sin^2\phi)^{3/2} = \dfrac{1 - e^2}{a^2} N^3$; $N = a/(1 - e^2 \sin^2\phi)^{1/2}$

Now from (25) and (26) it is seen that we will be able to express the quantity in brackets in terms of $\sec\phi$ and $\sec\phi_0$ since

$$\left(\frac{N}{R}\right)^2 = \frac{(1 - e^2 \sin^2\phi)^2}{(1 - e^2)^2} = \frac{[(1 - e^2) \sec^2\phi + e^2]^2}{(1 - e^2)^2 \sec^4\phi} \tag{27}$$

With the values of $(d\,\Delta\lambda)^2$ and $\left(\dfrac{N}{R}\right)^2$ from (25) and (27), the linear element (26) may be be written

$$ds^2 = \left[\sec^2\phi + \frac{[(1 - e^2) \sec^2\phi + e^2]^2}{(1 - e^2)^2 (\sec^2\phi_0 - \sec^2\phi)}\right] (R^2 \cos^2\phi d\phi^2). \tag{28}$$

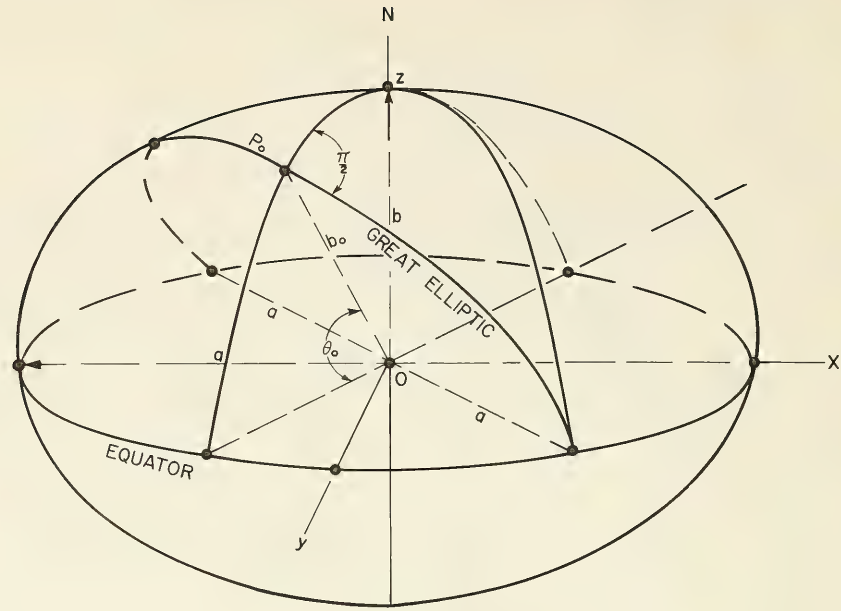
If the quantity in brackets is given a common denominator, then (28) may be written as

$$ds^2 = \frac{(1 - e^2) \sec^2\phi\, [(1 - e^2) \sec^2\phi_0 + 2e^2] + e^4}{(1 - e^2)^2 (\sec^2\phi_0 - \sec^2\phi)} \quad (R^2 \cos^2\phi d\phi^2)\ . \tag{29}$$

To bring (29) into manageable form we place $k = \dfrac{e\sqrt{1 - e^2}}{a} N_0 \sin\phi_0$, and (30)

$$\cos d = \frac{N \sin\phi}{N_0 \sin\phi_0}\ .$$

(Note that $k = e_0$, is the eccentricity of the great elliptic arc. See Figure 15.)



GREAT ELLIPTIC SECTION

Major semiaxis is a

Minor semiaxis is $b_o = a\sqrt{1-e^2 \sin^2\theta_o}$

a, e are semimajor axis and eccentricity of the ellipsoidal meridian

θ_o is the geocentric latitude of the vertex P_o of the Great Elliptic Section

e_o is the eccentricity of the Great Elliptic

$e_o = (a^2 - b_o^2)^{\frac{1}{2}}/a = e \sin\theta_o = (e\sqrt{1-e^2}/a) N_o \sin\phi_o$

Coordinates of P_o are $P_o\,(a \cos\theta_o \cos\lambda_o, a \cos\theta_o \sin\lambda_o, b \sin\theta_o)$ or in terms of geodetic latitude ϕ_o

$P_o\,(N_o \cos\phi_o \cos\Delta\lambda_o, N_o \cos\phi_o \sin\Delta\lambda_o, N_o(1-e^2)\sin\phi_o)$

Figure 15. Elements of the great elliptic section.

From the first of (30), placing $N_0 = a/(1 - e^2 \sin^2 \phi_0)^{1/2}$ and solving for $\sec^2\phi_0$ find

$$\sec^2\phi_0 = (1 - e^2 + k^2)/(1 - e^2)(1 - k^2/e^2). \tag{31}$$

With the value of $N_0 \sin \phi_0$ from the first of (30) placed in the second find $N \sin \phi = (ak/e\sqrt{1 - e^2}) \cos d$ and with $N = a/\sqrt{1 - e^2 \sin^2\phi}$, solving for $\sec^2\phi$ find

$$\sec^2\phi = \frac{1 - e^2 + k^2 \cos^2 d}{(1 - e^2)[1 - (k^2/e^2) \cos^2 d]}. \tag{32}$$

By differentiating $N \sin \phi = (ak/e\sqrt{1 - e^2}) \cos d$ obtain

$$(N \sin \phi)' d\phi = -(ak/e\sqrt{1 - e^2}) \sin d\, \delta d \tag{33}$$

Since $(N \sin \phi)' = \dfrac{R \cos \phi}{1 - e^2}$, equation (33) may be written

$$\frac{R \cos \phi}{1 - e^2} d\phi = -(ak/e\sqrt{1 - e^2}) \sin d\, \delta d \text{ or finally}$$

$$(R^2 \cos^2 \phi d\phi^2) = (1 - e^2) a^2 (k^2/e^2) \sin^2 d\, \delta d^2. \tag{34}$$

Now from (31) and (32) find

$$\sec^2\phi_0 - \sec^2\phi = \frac{(k^2/e^2) \sin^2 d}{(1 - e^2)(1 - k^2/e^2)[1 - (k^2/e^2) \cos^2 d]}, \tag{35}$$

and the numerator of (29) becomes

$$(1 - e^2) \sec^2\phi [(1 - e^2) \sec^2\phi_0 + 2e^2] + e^4 = \frac{1 - k^2 + k^2 \cos^2 d}{(1 - k^2/e^2)[1 - k^2/e^2) \cos^2 d]}. \tag{36}$$

With the values from (34), (35), (36) the linear element (29) becomes

$$ds^2 = \frac{1 - k^2 + k^2 \cos^2 d}{(1 - k^2/e^2)[1 - (k^2/e^2) \cos^2 d]} \cdot \frac{(1 - e^2)(1 - k^2/e^2)[1 - (k^2/e^2) \cos^2 d]}{(k^2/e^2) \sin^2 d\, (1 - e^2)^2} \cdot (1 - e^2).$$

$$a^2(k^2/e^2) \sin^2 d\, \delta d^2 = a^2(1 - k^2 + k^2 \cos^2 d)\, \delta d^2,$$

$$ds^2 = a^2(1 - k^2 \sin^2 d)\, \delta d^2. \tag{37}$$

Now equation (37) is the usual elliptic integral form with modulus k, and we write

$$s = a \left[\int_0^{d_1} + \int_0^{d_2}\right] (1 - k^2 \sin^2 d)^{1/2}\, \delta d, \tag{38}$$

where $k = (e\sqrt{1 - e^2}/a) N_0 \sin \phi_0$, the modulus of the elliptic integral, and $d_1 = \cos^{-1}(N_1 \sin \phi_1/N_0 \sin \phi_0)$, $d_2 = \cos^{-1}(N_2 \sin \phi_2/N_0 \sin \phi_0)$. (k is equal to e_0 the eccentricity of the great elliptic arc — see Figure 15).

The integrand of (38) may be expanded by the binomial formula and integrated term by term to obtain an approximation formula for direct computation. To 6th order terms in k: $(1 - k^2 \sin^2 d)^{1/2} = 1 - \tfrac{1}{2}k^2 \sin^2 d - (1/8)k^4 \sin^4 d - (1/16)k^6 \sin^6 d -.$ (39)

Making the identity substitutions

$\sin^2 d = ½ - ½ \cos 2d$, $\sin^4 d = (3/8) - ½\cos 2d + (\cos 4d)/8$

$\sin^6 d = (5/16) - (15/32)\cos 2d + (3/16)\cos 4d - (1/32)\cos 6d$, in (39) and integrating term by term according to (38) one obtains

$$s/a = (d_1 + d_2) - ½k^2[½(d_1 + d_2) - ¼(\sin 2d_1 + \sin 2d_2)] - (1/8)k^4[(3/8)(d_1 + d_2) - ¼(\sin 2d_1 + \sin 2d_2) + (1/32)(\sin 4d_1 + \sin 4d_2)] - (1/16)k^6[(5/16)(d_1 + d_2) - (15/64)(\sin 2d_1 + \sin 2d_2) + (3/64)(\sin 4d_1 + \sin 4d_2) - (1/192)(\sin 6d_1 + \sin 6d_2)]. \quad (40)$$

By means of the identity $\sin x + \sin y = 2\sin ½(x + y)\cos ½(x - y)$, equation (40) may be written finally as

$$s/a = (d_1 + d_2) - ¼k^2[(d_1 + d_2) - \sin(d_1 + d_2)\cos(d_1 - d_2)] - (1/128)k^4[6(d_1 + d_2) - 8\sin(d_1 + d_2)\cos d_1 - d_2) + \sin 2(d_1 + d_2)\cos 2(d_1 - d_2)] - (1/1536)k^6[30(d_1 + d_2) - 45\sin(d_1 + d_2)\cos(d_1 - d_2) + 9\sin 2(d_1 + d_2)\cos 2(d_1 - d_2) - \sin 3(d_1 + d_2)\cos 3(d_1 - d_2)], \quad (41)$$

a and e are semimajor axis and eccentricity of the meridian ellipse, $k = (e\sqrt{1 - e^2}/a)\, N_0 \sin\phi_0$ ($k = e_0$, the eccentricity of the great elliptic arc), ϕ_0 is the vertex of the great elliptic arc as given by (21). $d_1 = \arccos(N_1 \sin\phi_1/N_0 \sin\phi_0)$, $d_2 = \arccos(N_2 \sin\phi_2/N_0 \sin\phi_0)$. When $\phi_0 = 90°$; equation (41) gives a meridian arc of the spheroid. When $\phi_0 = 0$, an arc of the equator or circle of radius a is given. Formula (41) thus consists of a circular arc and successive corrective terms.

To examine the contribution of the terms in (41) take the case $\phi_1 = \phi_2 = 0$, $\phi_0 = 45°$, $d_1 = d_2 = 90°$ which will give the semilength of the great ellipse making an angle of 45° with the equator. For the Clarke 1866 spheroid, $e^2 = 6.768657997 \times 10^{-3}$, $a = 6{,}378{,}206.4$ meters. From (41) we have then

1st term $a \times (d_1 + d_2) = 20{,}037{,}773$ meters

2nd term $-a \times 2.65804 \times 10^{-3} = -16{,}954$ meters

3rd term $-a \times 0.17 \times 10^{-5} = -11$ meters

4th term $-a \times 0.24 \times 10^{-8} = -0.015$ meters

When $\phi_0 = 90$, $\phi_1 = \phi_2 = 0$, $d_1 + d_2 = \pi$, and (41) reduces to the usual formula for length of the semimeridian from equator to equator through the pole $s = a\pi[1 - ¼e^2 - (3/64)e^2 - (5/256)e^6 - - -]$.

GREAT ELLIPTIC ARC LENGTH IN TERMS OF PARAMETRIC LATITUDE θ

Equation (41) gives the arc length, but the modulus k, d_1 and d_2, and vertex ϕ_0 must be expressed in terms of parametric latitude, θ, if the geographic latitudes ϕ_1, ϕ_2 of the given points P_1, P_2 have been first converted to parametric latitudes θ_1, θ_2.

The relationships $\tan\phi = \dfrac{\tan\theta}{(1-e^2)^{1/2}}$, $N \sin\phi = \dfrac{a}{(1-e^2)^{1/2}} \sin\theta$, applied to

$k = (e\sqrt{1-e^2}/a)\, N_0 \sin\phi_0$,

$d_1 = \arccos(N_1 \sin\phi_1/N_0 \sin\phi_0)$, $d_2 = \arccos(N_2 \sin\phi_2/N_0 \sin\phi_0)$, and the last of equations (21) give

$e_0 = k = e\sin\theta_0$, $d_1 = \arccos(\sin\theta_1/\sin\theta_0)$, $d_2 = \arccos(\sin\theta_2/\sin\theta_0)$,

$\tan\theta_0 = (\tan^2\theta_1 + \tan^2\theta_2 - 2\tan\theta_1\tan\theta_2\cos\Delta\lambda)^{1/2}/\sin\Delta\lambda$,

whence

$$\sin\theta_0 = \tan\theta_0/(1+\tan^2\theta_0)^{1/2} , \tag{42}$$

$$\sin\theta_0 = \left(\frac{\tan^2\theta_1 + \tan^2\theta_2 - 2\tan\theta_1\tan\theta_2\cos\Delta\lambda}{\tan^2\theta_1 + \tan^2\theta_2 - 2\tan\theta_1\tan\theta_2\cos\Delta\lambda + \sin^2\Delta\lambda}\right)^{1/2} .$$

Equations (41) and (42) give then the arc length along the great elliptic arc when geographic latitudes have been converted to parametric latitudes.

THE CHORD DISTANCE

The chord distance between the points $Q_1(x_1, O, z_1)$, $Q_2(x_2, y_2, z_2)$ as shown in Figures (13) and (14) is given by the usual distance formula where the coordinates may be expressed in terms of either ϕ or θ, that is from (1)

$$x_1 = N_1\cos\phi_1,\ y_1 = 0,\ z_1 = N_1(1-e^2)\sin\phi_1 \text{ (in terms of } \phi)$$

$$x_2 = N_2\cos\phi_2\cos\Delta\lambda,\ y_2 = N_2\cos\phi_2\sin\Delta\lambda,\ z_2 = N_2(1-e^2)\sin\phi_2 , \tag{43}$$

or $$x_1 = a\cos\theta_1,\ y = 0,\ z_1 = a\sqrt{1-e^2}\ \sin\theta_1 \text{ (in terms of } \theta)$$

$$x_2 = a\cos\theta_2\cos\Delta\lambda,\ y_2 = a\cos\theta_2\sin\Delta\lambda,\ z_2 = a\sqrt{1-e^2}\sin\theta_2 .$$

Applying the distance formula to each set of formulas in (43) for coordinates one obtains

$$C = [(N_1\cos\phi_1 - N_2\cos\phi_2\cos\Delta\lambda)^2 + N_2^2\cos^2\phi_2\sin^2\Delta\lambda + (1-e^2)^2(N_1\sin\phi_1 - N_2\sin\phi_2)^2]^{1/2} \tag{44}$$

and in terms of θ

$$C = a[(\cos\theta_2\cos\Delta\lambda - \cos\theta_1)^2 + \cos^2\theta_2\sin^2\Delta\lambda + (1-e^2)(\sin\theta_2 - \sin\theta_1)^2]^{1/2} \tag{45}$$

In (45), expand the quantities in the brackets combining terms to obtain

$$C = a[2 - 2(\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2\cos\Delta\lambda) - e^2(\sin\theta_2 - \sin\theta_1)^2]^{1/2} . \tag{46}$$

Now $\cos(d_1 + d_2) = \sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2\cos\Delta\lambda$ and with $\sin\theta_1 = \sin\theta_0\cos d_1$, $\sin\theta_2 = \sin\theta_0\cos d_2$, $k^2 = e^2\sin^2\theta_0$ from (42), equation (46) can be written

$$C = a[2\{1 - \cos(d_1 + d_2)\} - k^2(\cos d_1 - \cos d_2)^2]^{1/2}. \tag{47}$$

With the identity $(\cos d_1 - \cos d_2)^2 = [1 - \cos(d_1 + d_2)]\,[1 - \cos(d_1 - d_2)]$,

we can write (47) finally as

$$C = a\left[\{1 - \cos(d_1 + d_2)\}\,\{2 - k^2[1 - \cos(d_1 - d_2)]\}\right]^{1/2} \quad . \tag{48}$$

Now (48) gives the chord length no matter which latitude is used, ϕ or θ, since for ϕ:

$d_1 = \arccos(N_1 \sin\phi_1 / N_0 \sin\phi_0)$, $d_2 = \arccos(N_2 \sin\phi_2 / N_0 \sin\phi_0)$,

$k^2 = [e^2(1 - e^2)/a^2)]\, N_0^2 \sin^2\phi_0$; while for θ:

$d_1 = \arccos(\sin\theta_1/\sin\theta_0)$, $d_2 = \arccos(\sin\theta_2/\sin\theta_0)$, $k^2 = e^2 \sin^2\theta_0$. Also (41) and (48) make it possible to prepare a computing form in terms of either ϕ or θ with corresponding azimuth forms from equations (12), (13), (15), (16), (17), (18).

THE ANGLE BETWEEN THE CHORD AND THE HORIZON AT A GIVEN POINT OF THE ELLIPSOID

Referring to Figure 13, it is seen that the angle β is determined by a perpendicular, u, from Q_2 upon the tangent at Q_1 and the chord c. That is $\sin B = u/c$.

Now the length of u is obtained by normalizing the equation of the tangent plane at Q_1, equation (4), and substituting the coordinates of the point Q_2 from (1):

$$u = \frac{1}{N_1}\,[a^2 - N_1 N_2 \cos\phi_1 \cos\phi_2 \cos\Delta\lambda - (1 - e^2)\, N_1 N_2 \sin\phi_1 \sin\phi_2]. \tag{49}$$

We can express u in parametric latitude, θ, since $(1 - e^2)\, N_1 N_2 \sin\phi_1 \sin\phi_2 = a^2 \sin\theta_1 \sin\theta_2$, $N_1 N_2 \cos\phi_1 \cos\phi_2 = a^2 \cos\theta_1 \cos\theta_2$, $N_1 = (a/\sqrt{1 - e^2})\sqrt{1 - e^2\cos^2\theta_1}$, i.e.

$$u = a\sqrt{1 - e^2}\;\frac{1 - (\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2 \cos\Delta\lambda)}{\sqrt{1 - e^2\cos^2\theta_1}} \tag{50}$$

Referring to equation (46) and the discussion there,

$\cos(d_1 + d_2) = \sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2 \cos\Delta\lambda$,

$\sin\theta_1 = \sin\theta_0 \cos d_1$, $k = e \sin\theta_0$ and (50) can be written in the form

$$u = b\,\frac{1 - \cos(d_1 + d_2)}{(1 - e^2 + k^2\cos^2 d_1)^{1/2}} \quad , \tag{51}$$

Where $b = a\sqrt{1 - e^2}$ is the minor semiaxis of the reference ellipsoid. From (48) and (51) we have then

$$\sin\beta = \frac{u}{c} = \left\{\frac{(1 - e^2)\,[1 - \cos(d_1 + d_2)]}{[2 - k^2\{1 - \cos(d_1 - d_2)\}]\,(1 - e^2 + k^2\cos^2 d_1)}\right\}^{1/2} \tag{52}$$

and thus $\sin\beta$ is expressed in the same quantities as the distance and chord lengths; see equations (41) and (48).

MAXIMUM SEPARATION OF CHORD AND ELLIPTIC ARC

In Figure 14, H_0 is the maximum separation between the great elliptic arc and the chord. As shown, this occurs when the tangent to the ellipse is parallel to the chord. Also when this occurs the center of the ellipse, the midpoint of the chord, and the point P on the curve are collinear, [10]. Hence the geographic coordinates of the point P can be found from the intersection of the meridian through Q and the plane of the great elliptic section.

The coordinates of Q, the midpoint of the chord Q_1Q_2, are

$$Q \begin{cases} (a/2)(\cos\theta_2 \cos\Delta\lambda + \cos\theta_1) \\ (a/2)(\cos\theta_2 \sin\Delta\lambda) \\ (b/2)(\sin\theta_1 + \sin\theta_2) \end{cases}$$

and the meridian through Q has the equation $(\cos\theta_2 \sin\Delta\lambda)\, x - (\cos\theta_1 + \cos\theta_2 \cos\Delta\lambda)\, y = 0.$ (53)

The equation to the plane of the great elliptic arc in terms of parametric latitude is

$$Ax + By + Cz = 0, \tag{54}$$

$$A = b\tan\theta_1 \sin\Delta\lambda,\quad B = b(\tan\theta_2 - \tan\theta_1\cos\Delta\lambda),\quad C = -a\sin\Delta\lambda$$

(Compare equation (14), where it is in terms of geodetic latitude ϕ). Now the point P ($a\cos\theta\cos\lambda$, $a\cos\theta\sin\lambda$, $b\sin\theta$) on the the ellipsoid must satisfy both equations (53) and (54) if it is to be the required point P on the great elliptic arc. This leads to the equations $\cos\theta_2 \sin\Delta\lambda\cos\lambda - (\cos\theta_1 + \cos\theta_2\cos\Delta\lambda)\sin\lambda = 0,$

$$A\cos\lambda + B\sin\lambda + C\tan\theta = 0, \tag{55}$$

where A, B, C are those of equation (54).

Solving (55) for λ and θ find,

$$P \begin{cases} \lambda = \arctan\left[(\cos\theta_2 \sin\Delta\lambda)/(\cos\theta_2\cos\Delta\lambda + \cos\theta_1)\right], \\ \theta = \arctan\left[\dfrac{(\tan\theta_1\sin\Delta\lambda)\cos\lambda + (\tan\theta_2 - \tan\theta_1\cos\Delta\lambda)\sin\lambda}{\sin\Delta\lambda}\right], \end{cases} \tag{56}$$

$$\theta = \arctan\left[\frac{\tan\theta_2\sin\lambda + \tan\theta_1\sin(\Delta\lambda - \lambda)}{\sin\Delta\lambda}\right]$$

$$\theta = \arctan\left[(\sin\theta_1 + \sin\theta_2)/(\cos^2\theta_1 + \cos^2\theta_2 + 2\cos\theta_1\cos\theta_2\cos\Delta\lambda)^{1/2}\right].$$

We have seen that

$$\begin{aligned} &\cos(d_1 + d_2) = \sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2\cos\Delta\lambda \\ &\sin\theta_1 = \sin\theta_0\cos d_1,\ \sin\theta_2 = \sin\theta_0\cos d_2 \end{aligned} \tag{57}$$

whence we can express

$$\cos^2\theta_1 + \cos^2\theta_2 + 2\cos\theta_1\cos\theta_2\cos\Delta\lambda = [1+\cos(d_1+d_2)][2-\sin^2\theta_0\{1+\cos(d_1-d_2)\}],$$

$$(\sin\theta_1 + \sin\theta_2)^2 = \sin^2\theta_0\,[1+\cos(d_1+d_2)]\,[1+\cos(d_1-d_2)]$$

and the last equation of (56) may be written

$$\theta = \arc\tan \frac{\sin\theta_0 \sqrt{1+\cos(d_1-d_2)}}{\sqrt{2-\sin^2\theta_0\,[1+\cos(d_1-d_2)]}} \,. \tag{58}$$

It is known that $H_0{}^2 = PP'^2$ will be given by $H_0{}^2 = [(y-y_1)r-(z-z_1)q]^2 + [(z-z_1)p - (x-x_1)r]^2 + [(x-x_1)q-(y-y_1)p]^2$, where x,y, z, are coordinates of P; x_1, y_1, z_1 are coordinates of Q_1 and p, q, r are direction cosines of the chord $c = Q_1Q_2$, [11]. See Figure 14. (59)

From (56) and (58) we can express the rectangular coordinates of P as

$$\text{P:}\quad x = a\cos\theta\cos\lambda = \frac{a}{\sqrt{2}}\;\frac{\cos\theta_1+\cos\theta_2\cos\Delta\lambda}{\sqrt{1+\cos(d_1+d_2)}}$$

$$y = a\cos\theta\sin\lambda = \frac{a}{\sqrt{2}}\;\frac{\cos\theta_2\sin\Delta\lambda}{\sqrt{1+\cos(d_1+d_2)}} \tag{60}$$

$$z = b\sin\theta = \frac{b}{\sqrt{2}}\;\frac{\sin\theta_1+\sin\theta_2}{\sqrt{1+\cos(d_1+d_2)}}$$

If the coordinates from (1) are converted to parametric latitude they will be Q_1 ($a\cos\theta_1$, O, $b\sin\theta_1$); Q_2 ($a\cos\theta_2\cos\Delta\lambda$, $a\cos\theta_2\sin\Delta\lambda$, $b\sin\theta_2$) whence the direction cosines of the chord $c = Q_1Q_2$ are

$$p = \frac{a}{c}(\cos\theta_2\cos\Delta\lambda - \cos\theta_1)$$

$$q = \frac{a}{c}\cos\theta_2\sin\Delta\lambda \tag{61}$$

$$r = \frac{b}{c}(\sin\theta_2 - \sin\theta_1)$$

From (60) and the coordinates of Q_1 ($a\cos\theta_1$, O, $b\sin\theta_1$) we have

$$x - x_1 = \frac{a}{\sqrt{2}R_0}(\cos\theta_1+\cos\theta_2\cos\Delta\lambda) - a\cos\theta_1$$

$$y - y_1 = (a\cos\theta_2\sin\Delta\lambda)/\sqrt{2}\,R_0 \tag{62}$$

$$z - z_1 = \frac{b}{\sqrt{2}\,R_0}(\sin\theta_1+\sin\theta_2) - b\sin\theta_1$$

Where $R_0 = \sqrt{1+\cos(d_1+d_2)} = \sqrt{2}\cos\tfrac{1}{2}(d_1+d_2)$.

With the values from (61) and (62) the expression (59) is formed to give

$$H_0{}^2 = \frac{a^2(\sqrt{2}-R_0)^2}{c^2 R_0{}^2}\cos^2\theta_1\cos^2\theta_2\left[b^2(\tan^2\theta_1+\tan^2\theta_2-2\tan\theta_1\tan\theta_2\cos\Delta\lambda)+a^2\sin^2\Delta\lambda\right] \tag{63}$$

Where $R_0 = [1 + \cos (d_1 + d_2)]^{1/2} = \sqrt{2} \cos \frac{1}{2}(d_1 + d_2)$.

Using the relationships (42), (48), (57) equation (63) can be solved for H_0 in any of the following several forms:

$$H_0 = \frac{b_0 (\sqrt{2} - \sqrt{1 + \cos (d_1 + d_2)}}{\sqrt{2 - k^2 \{1 - \cos(d_1 - d_2)\}}}, \tag{64}$$

$$= \frac{ab_0}{c} \left(\frac{(\sqrt{2}}{R_0} - 1 \right) \sin (d_1 + d_2),$$

$$= \frac{2ab_0}{c} \sin \tfrac{1}{2}(d_1 + d_2) [1 - \cos \tfrac{1}{2}(d_1 + d_2)],$$

Where $R_0 = \sqrt{1 + \cos (d_1 + d_2)} = \sqrt{2} \cos \frac{1}{2}(d_1 + d_2)$

$b_0 = \sqrt{1 - k^2} = a\sqrt{1 - e_0{}^2}$ = minor semiaxis of the great elliptic arc – see Figure 15. Thus H_0 is also expressed in quantities common with other elements of the great elliptic arc – see equations (41), (48), and (52).

A COMPUTING FORM FOR GREAT ELLIPTIC ARC LENGTH AND ASSOCIATED ELEMENTS

Since the computations to be discussed with the great elliptic arc approximation and the Andoyer-Lambert approximation both involve corrections to spherical elements, the basic spherical approximation is reviewed in Figure 16, and basic spherical formulae listed.

Now from (42) write

$$\sin^2 \theta_0 = K/(K + 1),$$

$$K = (A \tan \theta_1 + B \tan \theta_2)/ \sin^2 \Delta\lambda \tag{65}$$

$$A = \tan \theta_1 - \tan \theta_2 \cos \Delta\lambda, \; B = \tan \theta_2 - \tan \theta_1 \cos \Delta\lambda. \tag{66}$$

Azimuth equations (17) become

$$\cot \alpha_{AB} = D_1 (R_1 - B), \; \cot \alpha_{BA} = D_2 (A - R_2)$$

$$D_1 = \cos \theta_1 / T_1 \sin \Delta\lambda, \; D_2 = \cos \theta_2 / T_2 \sin \Delta\lambda \tag{67}$$

$$R_1 = C/\cos \theta_2, \; R_2 = - C/\cos \theta_1$$

$$C = e^2 (\sin \theta_2 - \sin \theta_1)$$

$$T_1 = (1 - e^2 \cos^2\theta_1)^{1/2}, \; T_2 = (1 - e^2 \cos^2\theta_2)^{1/2}$$

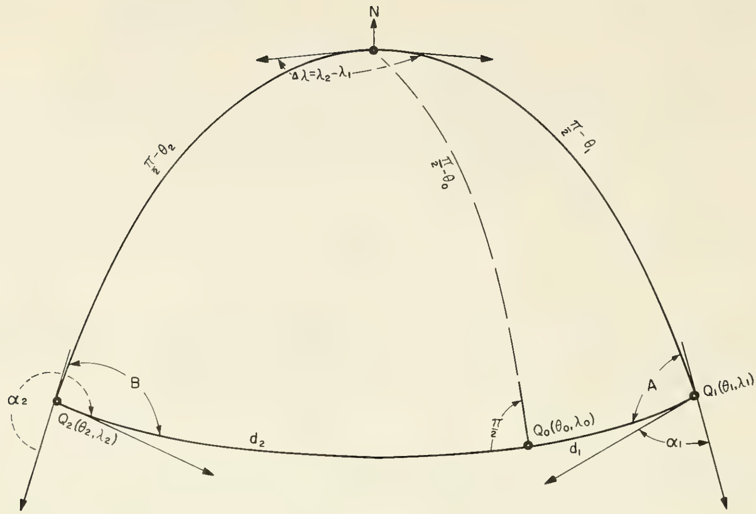
Equation (41) becomes

$$s = a (H + U_1 + U_2 + U_3) \tag{68}$$

where $U_1 = -N_1 (H - Q_1)$, $U_2 = - N_2 (6H - 8Q_1 + Q_2)$,

$$U_3 = - N_3 (30H - 45Q_1 + 9Q_2 - Q_3)$$

$$k^2 = e^2 \sin^2 \theta_0 = e_0{}^2 \text{ (eccentricity of the great elliptic arc).}$$



$$\cot A = \frac{\cos\theta_1 \tan\theta_2 - \sin\theta_1 \cos\Delta\lambda}{\sin\Delta\lambda}$$

$$\cot B = \frac{\cos\theta_2 \tan\theta_1 - \sin\theta_2 \cos\Delta\lambda}{\sin\Delta\lambda}$$

$$\cos(d_1 + d_2) = \sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2 \cos\Delta\lambda$$

$$\sin(d_1 + d_2) = (\cos\theta_1 \sin\Delta\lambda)/\sin B = (\cos\theta_2 \sin\Delta\lambda)/\sin A$$

$$\sin\theta_1 = \sin\theta_0 \cos d_1 , \quad \sin\theta_2 = \sin\theta_0 \cos d_2$$

NOTE: Q_0 may be external to Q_1Q_2, i.e. if either A or B is greater than 90°

Figure 16. Elements of polar spherical triangles.

$N_1 = k^2/4$, $N_2 = k^4/128 = 1/8\,N_1^2$, $N_3 = k^6/1536 = (1/3)\,N_1N_2$,

$Q_1 = \sin H \cos P$, $Q_2 = \sin 2H \cos 2P$, $Q_3 = \sin 3H \cos 3P$, $H = d_1 + d_2$, $P = d_1 - d_2$.

d_1 and d_2 are computed from

$$\cos 2d_1 = 2(1 - \cos^2\theta_1)/\sin^2\theta_0 - 1$$

$$\cos 2d_2 = 2(1 - \cos^2\theta_2)/\sin^2\theta_0 - 1 \qquad (69)$$

since $\cos^2\theta_1$ and $\cos^2\theta_2$ are already needed for T_1 and T_2, (67) above, and the use of $\sin^2\theta_0$ eliminates the computation of the square root of $K/(K + 1)$. A check is provided by $\sin(d_1 + d_2) = \sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2 \cos\Delta\lambda$.

From (48) the equation of the chord may be written

$$c = a(VW)^{1/2},\ V = (1 - \cos H),\ W = 2 - k^2R,\ R = (1 - \cos P). \qquad (70)$$

From (51) and (52) in terms of the symbols used above find

$$u = bV/T_1 \quad \sin\beta = bV/cT_1 = \frac{b}{T_1}\sqrt{\frac{V}{W}}\ . \qquad (71)$$

From (64) in terms of the above symbols find $H_0 = \dfrac{2ab_0}{c}(\sin \tfrac{1}{2}H)(1 - \cos \tfrac{1}{2}H)$, (72)

$b_0 = a\sqrt{1 - k^2}$, $k^2 = e^2 \sin^2\theta_0$.

Figure 17, shows equations (65) through (72) arranged for computing and a computation performed on the line Moscow to Cape of Good Hope. On the form find the geodetic distance, the normal section azimuths, the chord distance, the angle between the chord and the horizon at P_1, and the maximum separation of the chord and surface. The following table lists these values and gives a comparison with the distances computed by the rigorous Helmert method and the Andoyer-Lambert Approximation. Note that the geographic coordinates of the point $P(\phi,\lambda)$ where the maximum chord separation from the surface occurs may be computed from (56), (58), and already computed quantities in Figure (17).

MOSCOW TO CAPE OF GOOD HOPE

DISTANCE			AZIMUTHS		
Meters	n.m.	Method	Forward	Back	Type
10,102,069.91	5454.6814	Great Elliptic	15° 46' 56".744	190° 39' 27".350	Great Elliptic Section
			15° 49' 57".607	190° 41' 29".799	Normal Section
10,102,069.06	5454.6809	Helmert	15° 48' 17".674	190° 39' 32".208	Geodetic
10,102,065.28	5454.6789	Andoyer-Lambert	15° 48' 17".518	190° 39' 32".110	Geodetic

	meters	n.m.
CHORD DISTANCE	9,068,419.05	4896.5546
(MAXIMUM CHORD SEPARATION)	1,906,854.55	1029.6191
CHORD DEPRESSION ANGLE	45° 32' 37".462.	

Computations for distance, Normal Section Azimuths, Chord length, Angle of Depression of the Chord, Maximum Separation distance of chord and arc. Based on Great Elliptic Section Approximation to geodesic. Clarke 1866 Spheroid.

a = 6,378,206.4 meters, b = 6,356,583.8 meters, $e^2 = 6.7686580 \times 10^{-3}$, 1 radian = 206,264,8062 sec.

	° ' "				° ' "
ϕ_1	+55 45 19.500	1 (A)	Moscow	λ_1	−37 34 15.450
ϕ_2	−33 56 03.500	2 (B)	Cape of Good Hope	λ_2	−18 28 41.400
$\tan \phi_1$	+1.468 99522	$\tan \theta$	= 0.996609925 $\tan \phi$	$\delta\lambda = \lambda_2 - \lambda_1 =$	+19 05 34.050
$\tan \phi_2$	−0.672 84157			$\sin \delta\lambda$	+0.327 09901
$\tan \theta_1$	+1.464 01523	$\tan \theta_2$	−0.670 56059	$\cos \delta\lambda$	+0.944 99007
$\sin \theta_1$	+0.825 75246	$\sin \theta_2$	−0.556 93719	$\sin^2 \delta\lambda$	+0.106 99376
$\cos \theta_1$	+0.564 03269	$\cos \theta_2$	+0.830 55461	$T_1 = (1 - e^2 \cos^2\theta_1)^{1/2}$	+0.998 92275
$\cos^2 \theta_1$	+0.318 13288	$\cos^2 \theta_2$	+0.689 82096	$T_2 = (1 - e^2 \cos^2\theta_2)^{1/2}$	+0.997 66269

$A = \tan \theta_1 - \tan \theta_2 \cos \delta\lambda$ +2.097 68833 $\qquad D_1 = \cos \theta_1 / T_1 \ \sin \delta\lambda =$ +1.726 20806

$B = \tan \theta_2 - \tan \theta_1 \cos \delta\lambda$ −2.054 04044 $\qquad D_2 = \cos \theta_2 / T_2 \ \sin \delta\lambda =$ +2.545 10253

$K = (A \tan \theta_1 + B \tan \theta_2)/\sin^2 \delta\lambda$ +41.576 31463 $\qquad \sin^2 \theta_0 = K/(K+1) =$ +0.976 51276

$C = e^2(\sin \theta_2 - \sin \theta_1)$ −0.009 35895336 $\quad R_1 = C/\cos \theta_2$ −0.01126832 $\quad R_2 = -C/\cos \theta_2$ +0.0165 9293

$\cot \alpha_{(AB)} = D_1(R_1 - B)$ +3.526 2497 $\quad \alpha_{(AB)}$ 15° 49' 57.607" $\quad \cot \alpha_{(BA)} = D_2(A - R_2)$ +5.296 6012 $\quad \alpha_{(BA)}$ 190° 41' 29.799"

$\cos 2d_1 = 2(1 - \cos^2\theta_1)/\sin^2\theta_0 - 1$ +0.396 53499 $\quad d_1$ −33 19 08.864 $\quad H = d_1 + d_2$ +90 59 08.395

$\cos 2d_2 = 2(1 - \cos^2\theta_2)/\sin^2\theta_0 - 1$ −0.364 72093 $\quad d_2$ +124 18 17.259 $\quad P = d_1 - d_2$ −157 37 26.123

$\sin H$ +0.999 85203 $\quad \cos P$ −0.924 70507 $\quad \cos H$ −0.017 20226 $\quad$ H (radians) +1.587 99943

$\sin 2H$ −0.034 39942 $\quad \cos 2P$ +0.710 15891 $\quad k^2 = e^2 \sin^2\theta_0$ +6.6096809×10^{-3} $\quad N_1 = k^2/4$ +1.65242×10^{-3}

$\sin 3H$ −0.998 66853 $\quad \cos 3P$ −0.388 67002 $\quad N_2 = N_1^2/8$ +3.4131×10^{-7} $\quad N_3 = N_1 N_2/3$ +7.05×10^{-11}

$Q_1 = \sin H \cos P$ −0.924 56824 $\quad U_1 = -N_1(H - Q_1)$ −4.15182×10^{-3} $\quad V = 1 - \cos H$ +1.0172 0226

$Q_2 = \sin 2H \cos 2P$ −0.024 42905 $\quad U_2 = -N_2(6H - 8Q_1 + Q_2)$ −5.77×10^{-6} $\quad R = 1 - \cos P$ +1.924 70507

$Q_3 = \sin 3H \cos 3P$ +0.388 15252 $\quad U_3 = -N_3(30H - 45Q_1 + 9Q_2 - Q_3)$ −6.3×10^{-9} $\quad W = 2 - k^2R$ +1.987 278314

$\Sigma = H(\text{radians}) + U_1 + U_2 + U_3 =$ +1.583 8418 $\quad s = a\Sigma$ 10,102,069.91 meters 5454.6814 n.mi.

$VW =$ 2.0214640 $\quad (VW)^{1/2} =$ 1.42 17820 $\quad c = a(VW)^{1/2}$ 9,068,419.05 meters 4896.5546 n.mi.

$\sqrt{1-k^2}$ +.996 68968 $\quad b_0 = a\sqrt{1-k^2}$ 6,357,092.5 $\quad \sin \tfrac{1}{2}H =$ +.713 16276 $\quad \cos \tfrac{1}{2}H =$ +0.700 99848

$H_0 = (2a\, b_0/c)(\sin \tfrac{1}{2} H)(1 - \cos \tfrac{1}{2}H)$ 1,906,854.55 meters 1029.6191 n. mi.

$\sin \beta = bV/cT_1 =$ +.713 78531 $\qquad \beta$ 45° 32' 37.462"

Figure 17.

Figures 18 and 19 show the great elliptic arc formulae for distance arranged with geodetic azimuth formulae and the computations for distance and azimuth over the two lines (1) MOSCOW TO CAPE OF GOOD HOPE and (2) RAMEY AFB to MOUNTAIN HOME AFB.

No square roots are involved and only eight place tables of trigonometric functions, as Peters, are needed in addition to the constants for a particular spheroid of reference. The comparison with the Helmert rigorous and Andoyer-Lambert approximation is:

Line	Distance(meters)	Method	Forward Az.	Back Az.
(1)	10,102,069.91	Great Elliptic Arc	15° 48' 17".519	190° 39' 32".109
	10,102,069.06	Helmert	15° 48' 17".674	190° 39' 32".208
	10,102,065.28	Andoyer-Lambert	15° 48' 17".518	190° 39' 32".110
(2)	5,304,035.439	Great Elliptic Arc	131° 52' 34".985	285° 10' 06".870
	5,304,032.437	Helmert	131° 52' 35".29	285° 10' 06".65
	5,304,030.844	Andoyer-Lambert	131° 52' 35".043	285° 10' 06".869

REVIEW OF FORMER STUDIES

The Air Force Aeronautical Charting and Information Center made an extensive study of the Inverse Problem of Geodesy (1956–1957), over lines 50 to 6000 miles, [12]. A review of this study indicates favorably the use of the so called Andoyer-Lambert Formulae relative to requirements for Hyperbolic Electronic Systems since (1) they give very nearly geodetic distance with about the same error over all lines from 50 to at least 6000 miles, (2) azimuths are within about a second of true geodetic azimuths over all lines, (3) no tabular data for a particular spheroid is needed, (4) the only table of mathematical functions required is a table of the natural trigonometric functions as Peters eight place tables, (5) no root extraction is involved in the computations. The formulae are thus quite adaptable to small electric desk calculators or larger high speed digital machines. However, in review it seemed unnecessary to convert geographic coordinates to parametric before making the computations, hence a series of computations were made over the ACIC chosen lines for direct comparison. A representative group from 50 to 6000 miles was selected and additional comparisons were made against two lines whose true geodetic lengths and azimuths were known. No lines of 0° azimuth (meridional sections) were used because this is the trivial or limiting case and extensive tables of meridional distances for all reference ellipsoids are available or quite simple computation formulae are available for computing meridional arcs. The spherical formulae used are:

COMPUTATIONS, DISTANCE, AZIMUTHS

Great Elliptic Arc, Geodetic Azimuths

Clarke 1866 Ellipsoid; $a = 6{,}378{,}206.4$ meters, $e^2 = 6.6786580 \times 10^{-3}$,

$f/2 = 0.00169503765$, 1 radian = 206,264.8062 seconds, 1852 meters = 1 n. m.

ϕ_1 +55° 45′ 19.″500 — 1. (A) MOSCOW — λ_1 −37° 34′ 15.″450

ϕ_2 −33 56 03.500 — 2. (B) Cape of Good Hope — λ_2 − 18 28 41.400

$\tan\phi_1$ +1.468 99522 — 2. Always west of 1. — $\Delta\lambda = \lambda_2 - \lambda_1$ +19 05 34.050

$\tan\phi_2$ −0.672 84157 — $\tan\theta = 0.996609925 \tan\phi$ — $\sin\Delta\lambda$ +0.327 09901

$\tan\theta_1$ +1.464 01523 — $\tan\theta_2$ −0.670 56059 — $\cos\Delta\lambda$ +0.944 99007

$\sin\theta_1$ +0.82575246 — $\sin\theta_2$ −0.556 93719 — $\sin^2\Delta\lambda$ +0.106 99376

$\cos\theta_1$ +0.56403269 — $\cos\theta_2$ +0.830 55461 — $A = \tan\theta_1 - \tan\theta_2\cos\Delta\lambda$ +2.09768833

$\cos^2\theta_1$ +0.318 13288 — $\cos^2\theta_2$ +0.689 82096 — $B = \tan\theta_2 - \tan\theta_1\cos\Delta\lambda$ −2.05404044

$K = (A\tan\theta_1 + B\tan\theta_2)/\sin^2\Delta\lambda$ +41.576 31463 — $V_0 = \sin^2\theta_0 = K/(K+1)$ +0.976 51276

$\cos 2d_1 = 2(1-\cos^2\theta_1)/V_0 - 1$ +0.396 53499 — $\cos 2d_2 = 2(1-\cos^2\theta_2)/V_0 - 1$ −0.36472093

$H = d_1 + d_2$ +90° 59′ 08.″395 — H_r (radians) +1.58799943 — $P = d_1 - d_2$ −157° 37′ 26.″123 — $k^2 = e^2 V_0$ 6.6096809×10^{-3}

$\sin H$ +0.99985203 — $\cos P$ −0.924 70507 — $Q_1 = \sin H \cos P$ −0.92456824 — $N_1 = k^2/4$ $+1.65242 \times 10^{-3}$

$\sin 2H$ −0.034 39942 — $\cos 2P$ +0.710 15891 — $Q_2 = \sin 2H \cos 2P$ −0.024 42405 — $N_2 = N_1^2/8$ $+3.4131 \times 10^{-7}$

$\sin 3H$ −0.998 66853 — $\cos 3P$ −0.38867002 — $Q_3 = \sin 3H \cos 3P$ +0.388 15252 — $N_3 = N_1 N_2/3$ $+7.05 \times 10^{-11}$

$U_1 = -N_1(H_r - Q_1)$ -4.15182×10^{-3} — $U_2 = -N_2(6H_r - 8Q_1 + Q_2)$ -5.77×10^{-6} — $U_3 = -N_3(30H_r - 45Q_1 + 9Q_2 - Q_3)$ -6.3×10^{-9}

$\Sigma = H_r + U_1 + U_2 + U_3$ +1.583 8418 — $s = a\Sigma$ 10,102,069.91 meters 5454.6814 n. m.

$\cot A_0 = B\cos\theta_1/\sin\Delta\lambda$ −3.5418816 — $P_0'' = \dfrac{f}{2}\cdot\dfrac{H''}{\sin H}$ 555.″289 — $\cot B_0 = A\cos\theta_2/\sin\Delta\lambda$ +5.326 3528

A_0 164° 14′ 01.″416 — $\sin 2A_0$ −.522 98282 — $\sin 2B_0$ +0.362 70662 — B_0 10° 37′ 59.″721

δA_0 + 02 18.935 — $\delta A_0'' = P_0''\cos^2\theta_2 \sin 2B_0$ +138.″935 — δB_0 − 01 32.388

$-(A_0 - \delta A_0)$ −164 11 42.481 — $\delta B_0'' = P_0''\cos^2\theta_1 \sin 2A_0$ −92.″388 — $B_0 - \delta B_0$ 10° 39 32.109

$\alpha_{AB} = 180° - (A_0 - \delta A_0)$ 15° 48′ 17.″519 — $\alpha_{BA} = 180° + B_0 - \delta B_0$ 190° 39′ 32.″109

Figure 18.

COMPUTATIONS, DISTANCE, AZIMUTHS

Great Elliptic Arc, Geodetic Azimuths

Clark 1866 Ellipsoid: a = 6,378,206.4 meters, $e^2 = 6.7686580 \times 10^{3}$

f/2 = 0.00169503765, 1 radian = 206,264.8062 seconds, 1852 meters = 1 n. m.

	°	'	"				°	'	"
ϕ_1	18	29	57.9	1 (A)	Ramey Air Force Base	λ_1	67	07	30.3
ϕ_2	43	03	19.6	2 (B)	Mountain Home AFB	λ_2	115	52	54.7
					2. Always west of 1.	$\Delta\lambda = \lambda_2 - \lambda_1$	48	45	24.4

$\tan \phi_1$ +0.334 58400 $\quad$ $\tan \theta = 0.99660 9925 \tan \phi$ $\quad$ $\sin \Delta\lambda$ +0.751 91780

$\tan \phi_2$ +0.934 325 90 $\quad$ $\tan \theta_2$ +0.93115847 $\quad$ $\cos \Delta\lambda$ +0.659 25687

$\tan \theta_1$ +0.333 44994 $\quad$ $\sin \theta_2$ +0.68146713 $\quad$ $\sin^2 \Delta\lambda$ +0.565 38038

$\cos \theta_1$ +0.948 65017 $\quad$ $\cos \theta_2$ +0.73184872 $\quad$ $M = \tan\theta_2 - \tan\theta_1 \cos \Delta\lambda$ +.711 32944

$\sin \theta_1$ +0.316 32716

$\cos^2 \theta_1$ 0.899 93715 $\quad$ $\cos^2 \theta_2$ +0.53560255 $\quad$ $N = \tan\theta_1 - \tan\theta_2 \cos \Delta\lambda$ −.280 42288

$K = (N \tan \theta_1 + M \tan \theta_2)/\sin^2 \Delta\lambda$ 1.006 14296 $\quad$ $V_0 = \sin^2 \theta_0 = K/(K+1)$ +0.50153104

$\cos (d_1 + d_2) = \sin \theta_1 \sin \theta_2 + \cos \theta_1 \cos \theta_2 \cos \Delta\lambda$ +.67326778 $\quad$ $\cot A = M \cos \theta_1 / \sin \Delta\lambda$ +.897 44224

$\sin (d_1 + d_2) = \cos \theta_1 \sin \Delta\lambda / \sin B = \cos \theta_2 \sin \Delta\lambda / \sin A$ +.739 39875 $\quad$ $\cot B = N \cos \theta_2 / \sin \Delta\lambda$ −.272 93825

$\cos 2d_1 = 2 \sin^2 \theta_1 / V_0 - 1$ −.60097037 $\quad$ $\cos 2d_2 = 2 \sin^2 \theta_2 / V_0 - 1$ +0.85191908 $\quad$ A 48° 05' 37".885

d_1 and d_2 are always in the first or second quadrant. If A > 90°, $|d_2| > |d_1|$, $d_2 > 0$, $d_1 < 0$. $\quad$ B 105° 15' 58".929

If B > 90°, $|d_1| > |d_2|$, $d_1 > 0$, $d_2 < 0$.

$2d_1$ 126° 56' 21".938 $\quad$ $2d_2$ −31° 34' 44".339 $\quad$ $H = d_1 + d_2$ 47° 40' 48".800 $\quad$ $P = d_1 - d_2$ 79° 15' 33".139

$\sin H$ +.739 39875 $\quad$ $\cos P$ +.186 36618 $\quad$ $Q_1 = \sin H \cos P$ +.137 79892 $\quad$ H_r (radians) +.832 17687

$\sin 2H$ +.995 626 70 $\quad$ $\cos 2P$ −.930 53528 $\quad$ $Q_2 = \sin 2H \cos 2P$ −.926 46577 $\quad$ $k^2 = e^2 V_0$ $3.394\ 6921 \times 10^{-3}$

$\sin 3H$ +.601 24803 $\quad$ $\cos 3P$ −.53320682 $\quad$ $Q_3 = \sin 3H \cos 3P$ −.320 58955 $\quad$ $N_1 = k^2/4$ $.84867303 \times 10^{-3}$

$U_1 = -N_1(H_r - Q_1)$ $-.589\ 29984 \times 10^{-3}$ $\quad$ $U_2 = -N_2(6H_r - 8Q_1 + Q_2)$ $-.2669 \times 10^{-6}$ $\quad$ $N_2 = N_1^2/8$ 9.003×10^{-8}

$U_3 = -N_3(30H_r - 45Q_1 + 9Q_2 - Q_3)$ $-.274 \times 10^{-9}$ $\quad$ $N_3 = N_1 N_2 / 3$ 2.55×10^{-11}

$\Sigma = H_r + U_1 + U_2 + U_3$ +.831 58730 $\quad$ $s = a\Sigma$ 5,304,035.439 meters 2863.9500 n.m.

$T = (f/2)\ H''/\sin H$ 3 93.497 $\quad$ $\sin 2A$ +.994 17410 $\quad$ $\sin 2B$ −.508 3061

$\delta A'' = T \cos^2 \theta_2 \sin^2 B$ −107."130

$\delta B'' = T \cos^2 \theta_1 \sin 2A$ +352."059

	°	'	"		°	'	"
A	48	05	37.885	B	105	15	58.929
δA	−	01	47.130	δB +		5	52.059
$(A - \delta A)$	48	07	25.015	$(B - \delta B)$	105	10	06.870
$\alpha_{AB} = 180° - (A - \delta A)$	131	52	34.985	$\alpha_{BA} = 180° + (B - \delta B)$	285	10	06.870

Figure 19.

Spherical Formulae (see Figure 16)

$$\begin{aligned}
&\cos d = \sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2 \cos \Delta\lambda \\
&\sin A = (\cos \phi_2 \sin \Delta\lambda)/\sin d, \quad \sin B = (\cos \phi_1 \sin \Delta\lambda)/\sin d \\
&\cot A = (\cos \phi_1 \tan \phi_2 - \sin \phi_1 \cos \Delta\lambda)/\sin \Delta\lambda \\
&\cot B = (\cos \phi_2 \tan \phi_1 - \sin \phi_2 \cos \Delta\lambda)/\sin \Delta\lambda \\
&\sin d = (\cos \phi_1 \sin \Delta\lambda)/\sin B = (\cos \phi_2 \sin \Delta\lambda)/\sin A.
\end{aligned} \tag{73}$$

The Andoyer-Lambert correction [13] for distance is:

$$\delta d = -\frac{f}{4}\left[\frac{d + 3 \sin d}{1 - \cos d}(\sin \phi_1 - \sin \phi_2)^2 + \frac{d - 3 \sin d}{1 + \cos d}(\sin \phi_1 + \sin \phi_2)^2\right], \tag{74}$$

where d is spherical distance from (73) and $s = a(d + \delta d)$, f is the flattening, $f = (a - b)/a$, where a, b are the semiaxes of the reference ellipsoid (a is the radius of the auxiliary sphere).

Now (73) and (74) are essentially the same as used for several years in Loran computations except for the conversion to parametric latitudes which is not required with these formulas. The only difference in the appearance of the formulas is in the term 3 sin d in (74) which is simply sin d in the formulae for parametric latitude, [14].

The corrections to the spherical angles A and B as given by (73) to get geodesic azimuths are, [13]:

$$\begin{aligned}
\delta A &= \frac{f}{2}\left[\frac{d}{\sin d}\cos^2\phi_2 \sin 2B - \cos^2\phi_1 \sin 2A\right], \\
\delta B &= \frac{f}{2}\left[\cos^2\phi_2 \sin 2B - \frac{d}{\sin d}\cos^2\phi_1 \sin 2A\right],
\end{aligned} \tag{75}$$

the geodetic azimuths being then

$$\alpha_{AB} = 180° - A + \delta A, \quad \alpha_{BA} = 180 + B + \delta B.$$

The formulae as given by (73), (74), (75) were arranged in computing forms to make the check computations of the ACIC chosen lines. Note that the azimuths as given in the ACIC publications differ by 180° from the usual geodetic azimuths and the forward and back azimuths are interchanged from the conventions used in the check computations. The lines chosen are shown in TABLE 1, the comparisons are given in TABLES 2 and 3, while the actual computations are in Appendix 2.

TABLE 1

LINES COMPUTED

Line No.	Az.	Terminus Lat.	Terminus Long	Origin Lat.	Origin Long.	Distance Miles
	°	° ′ ″	° ′ ″	° ′ ″	° ′ ″	
1	45	40	18	40 30 37.757	17 19 43.280	50
2	90	10	18	9 59 48.349	16 31 55.877	100
3	90	70	18	69 48 05.701	9 37 28.637	200
4	45	10	18	13 04 12.564	14 51 13.283	300
5	45	70	18	73 35 09.206	3 26 35.101	400
6	90	40	18	39 37 06.613	8 36 43.276	500
7	45	40	18	44 54 28.507	10 47 43.883	500
8	45	70N	18W	76 00 26.603N	28 42 03.567E	1000
9	90	40N	18W	27 49 42.130N	32 54 12.997E	3000
10	45	40N	18W	35 18 45.644N	102 02 29.370E	6000
11	50	43 03 19.6	115 52 54.7	18 29 57.9	67 07 30.3	3000 n.m.
12	10	33 56 03.5S	18 28 41.4E	55 45 19.5N	37 34 15.450E	5500 n.m.

1–10 From ACIC Reports 59 (page 39), 80 (page 23).

11 Ramey AFB to Mountain Home AFB, AFAC–TN–57–53, Astia Document 135972, 1957

12 Cape of Good Hope to Moscow

TABLE 2

Comparison With True Distances and Azimuths

Line No.	Computed Distance S_c meters	True Distance S_t meters	$S_c - S_t = \Delta S$ meters	Computed α^c_{AB} ° ' "	True α^t_{AB} ° ' "	$\alpha^c_{AB} - \alpha^t_{AB} = \Delta\alpha_{AB}$	Computed α^c_{BA} ° ' "	True α^t_{BA} ° ' "	$\alpha^c_{BA} - \alpha^t_{BA} = \Delta\alpha_{BA}$ "
1	80,467.388	80,466.490	+0.898	45 26 00.443	45 26 01.692	–1.249	244 59 58.759	244 59 59.997	–1.238
2	160,935.945	160,932.956	+2.989	90 15 17.506	90 15 17.480	+0.026	270 00 00.023	270 00 00.000	+0.023
3	321,862.977	321,866.796	–3.819	97 52 01.112	97 52 01.063	+0.049	270 00 00.026	269 59 59.950	+0.076
4	482,794.743	482,798.163	–3.420	45 37 44.972	45 37 46.111	–1.139	224 59 58.629	224 59 59.732	–1.103
5	643,728.709	643,732.429	–3.720	58 50 30.885	58 50 31.600	–0.715	224 59 59.601	225 00 00.154	–0.553
6	804,664.697	804,664.762	–0.065	96 01 06.689	96 01 06.640	+0.049	270 00 00.073	270 00 00.001	+0.072
7	804,666.623	804,664.771	+1.861	49 52 14.352	49 52 15.528	–1.176	224 59 58.828	224 59 59.994	–1.166
8	1,609,315.609	1,609,329.060	–13.451	89 55 22.643	89 55 22.833	–0.190	224 59 59.834	224 59 59.958	–0.124
9	4,827,983.105	4,827,984.247	–1.142	119 54 41.396	119 54 41.260	+0.136	269 59 59.612	270 00 00.121	–0.509
10	9,655,972.218	9,655,969.751	+2.467	138 23 42.394	138 23 42.755	–0.361	225 00 00.674	225 00 00.276	+0.398
11	5,304,028.110	5,304,032.437	–4.327	131 52 35.913	131 52 35.290	+0.623	285 10 07.272	285 10 06.650	+0.622
12	10,102,057.97	10,102,069.06	–11.09	15 48 16.939	15 48 17.674	–0.735	190 39 31.445	190 39 32.208	–0.753

TABLE 3

Error Summary

Line No.	Azimuth	Terminal Latitude	S = distance		ΔS		Relative distance error $\Delta S_m/S_m$	$\Delta\alpha_{AB} = \Delta\alpha_{1-2}$	$\Delta\alpha_{BA} = \Delta\alpha_{2-1}$
	degrees	degrees	meters S_m	n.m.	meters ΔS_m	feet	1 part in	seconds	seconds
1	45	40N	80,466	43.5	+ 0.9	+ 3.0	89,407	− 1.25 **	− 1.24 **
2	90	10N	160,933	86.9	+ 3.0	+10.0	53,644	+ 0.03	+ 0.02
3	90	70N	321,867	173.8	− 3.8	+12.5	84,702	+ 0.05	+ 0.08
4	45	10N	482,798	260.7	−3.4	−11.2	141,899	− 1.14	− 1.10
5	45	70N	643,732	347.6	− 3.7	−12.2	173,982	− 0.72	− 0.55
6	90	40N	804,665	434.5	− 0.07	− 0.2	11,495,214	+ 0.05	+ 0.07
7	45	40N	804,667	434.5	+ 1.9	+ 6.0	423,509	− 1.18	− 1.17
8	45	70N	1,609,329	869.0	−13.5 *	−44.6	119,210	− 0.19	− 0.12
9	90	40N	4,827,984	2606.9	− 1.1	− 3.6	4,389,076	+ 0.14	− 0.51
10	45	40N	9,655,970	5213.8	+ 2.5	+ 8.2	3,862,388	− 0.36	+ 0.40
11	50	43N	5,304,032	2863.9	− 4.3	−14.2	1,233,496	+ 0.62	+ 0.62
12	10	34S	10,102,069	5454.7	−11.1	−36.6	910,096	− 0.74	− 0.75

* Maximum distance error

** Maximum azimuth errors

INVESTIGATION OF HIGHER ORDER TERMS IN ANDOYER-LAMBERT APPROXIMATION

While either form of Andoyer-Lambert approximation is probably satisfactory in the "state of the art" in hyperbolic navigational systems development, the question arises as to the higher order terms in the flattening of the Andoyer-Lambert approximation and the possibility of a single set of formulae which will give distance within one meter and azimuth within one second over all geodetic lines on the spheroid. This would be a practical operational system particularly if it maintained the several attributes of the Andoyer-Lambert first order approximation.

HISTORICAL

Now Lambert, [13], never published his derivation but had equivalent formulae for a first order approximation several years before the publication posthumously in 1932 of Andoyer's sketch, [15], of the derivation of the form as given in equation (74). Andoyer's derivation employs a differential oblique spherical triangle and it is not clear how one would proceed to higher order terms in the flattening. It is believed that Andoyer's derivation is the only recognized published one in existence.

DERIVATION FROM THE GREAT ELLIPTIC ARC

Independent derivations of the Andoyer-Lambert approximations were sought in the hopes of discovering a simple method of arriving at higher order terms in the flattening. It was noticed that the computations using the Andoyer-Lambert approximations; the ratios $(d - \sin d)/(1 + \cos d)$, $(d + \sin d)/(1 - \cos d)$ were being used in forming computational parameters, [16]. It was decided to try the ratios

$$(\sin \theta_1 + \sin \theta_2)^2/(1 + \cos d), (\sin \theta_1 - \sin \theta_2)^2/(1 - \cos d) \tag{76}$$

with the hope of relating these to other parameters and identification of the Andoyer-Lambert approximations in some other extant series expansion as the great elliptic arc approximation. See equations (19) through (42).

From equations (42) we have

$$\sin \theta_1 = \sin \theta_0 \cos d_1, \ \sin \theta_2 = \sin \theta_0 \cos d_2. \tag{77}$$

From (77), by simple algebraic operations and trigonometric identities, we may express (76) as

$$(\sin \theta_1 + \sin \theta_2)^2/(1 + \cos d) = 2 \sin^2 \theta_0 \cos^2 \tfrac{1}{2}(d_1 + d_2)$$

$$(\sin \theta_1 - \sin \theta_2)^2/(1 - \cos d) = 2 \sin^2 \theta_0 \sin^2 \tfrac{1}{2}(d_1 + d_2), \tag{78}$$

where $d = d_2 - d_1$.

From (78) by adding and subtracting respective members, we write

$$X = \frac{(\sin\theta_1 + \sin\theta_2)^2}{1 + \cos d} + \frac{(\sin\theta_1 - \sin\theta_2)^2}{1 - \cos d} = 2\,[\sin^2\theta_0] \qquad (79)$$

$$Y = \frac{(\sin\theta_1 + \sin\theta_2)^2}{1 + \cos d} - \frac{(\sin\theta_1 - \sin\theta_2)^2}{1 - \cos d} = 2[\sin^2\theta_0 \cos(d_1 + d_2)],$$

where $d = d_2 - d_1$.

The Andoyer-Lambert forms can now be written in terms of X and Y of (79) as

$$S = a[d - (f/4)(Xd - Y \sin d)],$$

$$S = a[d - (f/4)(Xd - 3Y \sin d)], \qquad (80)$$

where in the second equation, the geodetic latitude, ϕ, is used in forming the X and Y of (79).

If in the expansion of the great elliptic arc, equation (41), we place d_1 = to $-d_1$, and then $d = d_2 - d_1$, $k = e \sin\theta_0$, we obtain as far as sixth order terms in e:

$$S = a\left[\begin{array}{l} d - \frac{1}{4} e^2 \sin^2\theta_0 [d - \sin d \cos(d_1 + d_2)] \\ - (1/128)e^4 \sin^4\theta_0 [6d - 8 \sin d \cos(d_1 + d_2) + \sin 2d \cos 2(d_1 + d_2)] \\ - (1/1536)e^6 \sin^6\theta_0 \left[\begin{array}{l} 30d - 45 \sin d \cos(d_1 + d_2) + 9 \sin 2d \cos 2(d_1 + d_2) \\ \qquad - \sin 3d \cos 3(d_1 + d_2) \end{array}\right] \end{array}\right] \qquad (81)$$

Using relations (79), equation (81) can be written:

$$S = a\left[\begin{array}{l} d - (e^2/8)(Xd - Y \sin d) \\ - (e^4/512)[(6d - \sin 2d) X^2 - 8(\sin d) XY + 2(\sin 2d) Y^2] \\ - (e^6/12{,}288) \left[\begin{array}{l} 3(10d - 3 \sin 2d) X^3 - 3(15 \sin d - \sin 3d) X^2Y \\ \qquad + 18(\sin 2d) XY^2 - 4(\sin 3d) Y^3 \end{array}\right] \end{array}\right] \qquad (82)$$

Note in (82) that if all terms above the first power in f are ignored ($e^2 = 2f$) equation (82) reduces directly to the Andoyer-Lambert form as given by the first of (80). Now it is known that the difference in lengths of the great elliptic arc and the geodesic is of 4th order in e, [17], but the 6th order term will be useful for comparison later in the investigation.

DERIVATION FROM MODIFIED DIFFERENTIAL EQUATIONS

The corresponding differential triangles, auxiliary sphere, spheroid, where geodetic latitude has been converted to parametric are, as abstracted from Figure (20):

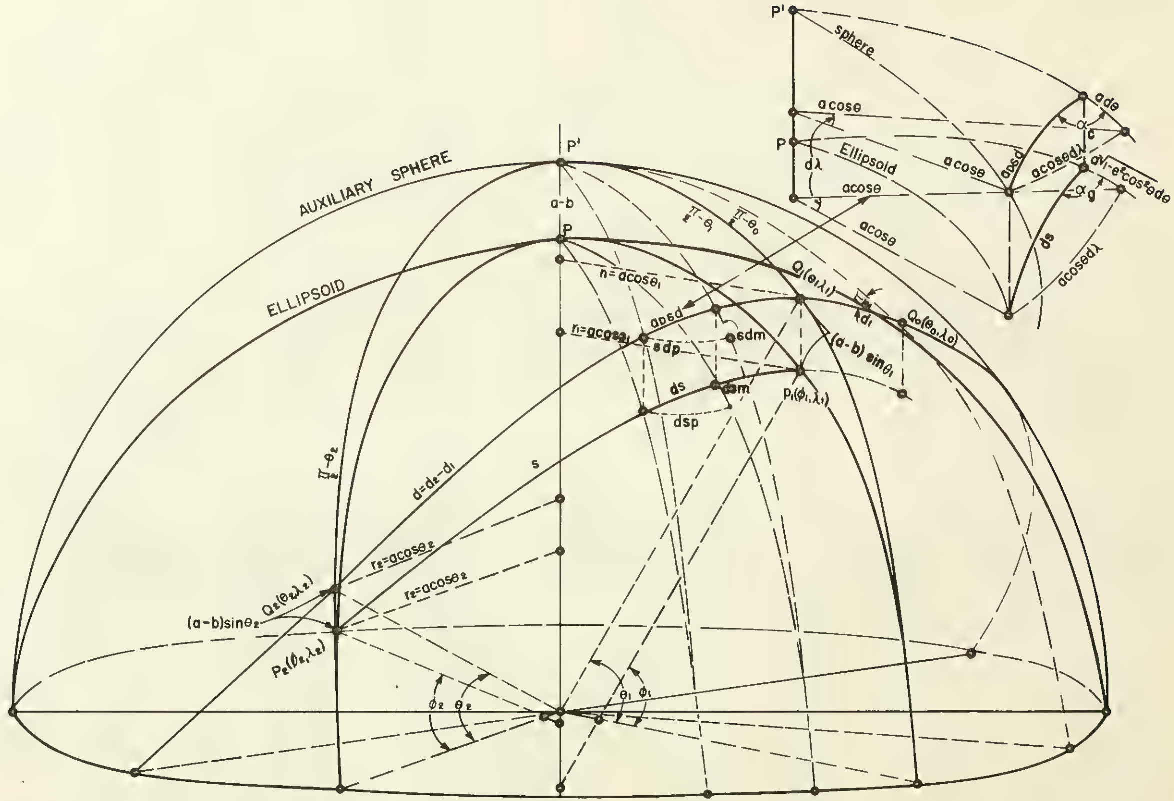
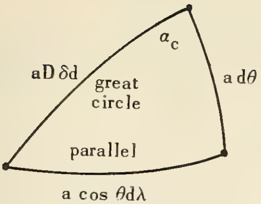
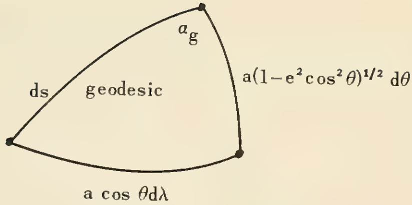


Figure 20. Differential triangles, sphere and spheroid.





and since $\alpha_c = \alpha_g$ (property of geodesics on surfaces of revolution, i. e. r sin α_c = r sin α_g, r = a cos θ), ds/aDδd = a$(1 - e^2 \cos^2\theta)^{1/2}$ dθ/adθ = $(1 - e^2 \cos^2\theta)^{1/2}$, which may be written

$$S = a(d + \delta d) = a\left[d + \int_{d_1}^{d_2}[(1 - e^2 \cos^2\theta)^{1/2} - 1]\, D\delta d\right]. \tag{83}$$

If (83) also represents the equator, then $\delta d = 0$, when $\theta = \theta_0 = 0$. Hence we add to the integrand $1 - (1 - e^2 \cos^2\theta_0)^{1/2}$ to get

$$S = a(d + \delta d) = a\left[d + \int_{d_1}^{d_2}\left[(1 - e^2 \cos^2 \theta)^{1/2} - (1 - e^2 \cos^2\theta_0)^{1/2}\right] D\delta d\right], \tag{84}$$

and we note that when $\theta = \theta_0 = 0$, $\delta d = 0$; when $\theta = \theta_0$, s = d = δd = 0; when $\theta_0 = \pi/2$, $d_1 = \theta_1$, $d_2 = \theta_2$, Dδd = dθ, d = $\theta_2 - \theta_1$ then (84) represents the meridian.

Expanding (84) to 6th order terms in e, find

$$S = a \left[\begin{array}{l} d - (e^2/2)(1 + e^2/2 + 3e^4/8) \int_{d_1}^{d_2} (\sin^2\theta_0 - \sin^2\theta)\, D\delta d \\ \quad + (e^4/8)(1 + 3e^2/2) \int_{d_1}^{d_2} (\sin^4\theta_0 - \sin^4\theta)\, D\delta d \\ \quad - (e^6/16) \int_{d_1}^{d_2} (\sin^6\theta_0 - \sin^6\theta)\, D\delta d \end{array}\right] \tag{85}$$

Now from (77), $\sin\theta = \sin\theta_0 \cos d$,

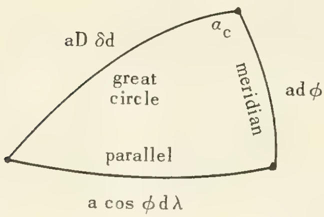
$$\sin^2\theta = \sin^2\theta_0 \cos^2 d = \frac{\sin^2\theta_0}{2}(1 + \cos 2d). \tag{86}$$

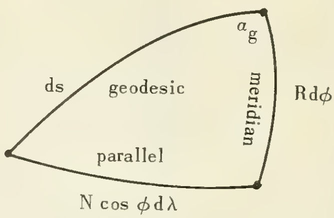
The value of $\sin^2\theta$ from (86) placed in (85) and the resulting integrations performed with respect to d, leads to expressions in powers of the right hand quantities in (79) so that (85) may be written finally as

$$S = a \left[\begin{array}{l} d - (e^2/8)\,(1 + e^2/2 + 3e^4/8)\,(Xd - Y \sin d) \\ \quad - (e^4/512)\,(1 + 3e^2/2) \left[\begin{array}{l} -(10d + \sin 2d)\,X^2 + 8(\sin d)\,XY \\ \qquad + 2(\sin 2d)\,Y^2 \end{array} \right] \\ \quad - (e^6/12{,}288) \left[\begin{array}{l} 3(22d + 3 \sin 2d)\,X^3 - 3(15 \sin d - \sin 3d)\,X^2Y \\ \qquad - 18(\sin 2d)\,XY^2 - 4(\sin 3d)Y^3 \end{array} \right] \end{array} \right] \tag{87}$$

Again if all terms above first order in f ($e^2 = 2f$) in (87) are ignored then the first two terms of (87) represent the Andoyer-Lambert form as given by the first of equations (80).

For the case where geographic latitudes, ϕ, are not first converted to parametric, but are considered spherical, the corresponding differential right triangles are:





We have for the approximation

$$Rd\phi = ds \cos \alpha_g$$

or $\quad Rd\phi = ds \dfrac{d\phi}{D\delta d}$, placing $\cos \alpha_g = \cos \alpha_c = \dfrac{d\phi}{D\delta d}$.

$$ds = R\,D\delta d = a(1 - e^2)\,(1 - e^2 \sin^2\phi)^{-3/2}\,D\delta d. \tag{88}$$

If (88) represents the equator, then when $\phi = 0$, $ds = aD\delta d$. Hence add $e^2 \cos^2\phi_0$ to the integrand of (88), to obtain

$$(ds/a) = [1 - e^2)\,(1 - e^2 \sin^2\phi)^{-3/2} + e^2 \cos^2\phi_0]\,D\delta d. \tag{89}$$

Note the following for (89): When $\phi = \phi_0 = 0$, $ds = a\,D\delta d$; when $\phi_0 = \pi/2$, $D\delta d = d\phi$, equation (89) will represent the meridian.

Expanding (89) to 6th order terms in e get

$$(ds/a) = \left[\begin{array}{l} 1 + (3/2)e^2 \sin^2\phi + (15/8)e^4 \sin^4\phi + (35/16)e^6 \sin^6\phi) \\ - e^2[1 + (3/2)e^2 \sin^2\phi + (15/8)e^4 \sin^4\phi] + e^2(1 - \sin^2\phi_0) \end{array} \right] D\delta d \tag{90}$$

which may be written in the integral form

$$S = a \left[\begin{aligned} & d - (e^2/2) \int_{d_1}^{d_2} (2 \sin^2\phi_0 - 3 \sin^2\phi)\, D\delta d \\ & - (3e^4/8) \int_{d_1}^{d_2} \sin^2\phi (4 - 5 \sin^2\phi)\, D\delta d \\ & - (5e^6/16) \int_{d_1}^{d_2} \sin^4\phi (6 - 7 \sin^2\phi)\, D\delta d \end{aligned} \right] \tag{91}$$

From (77), with θ replaced by ϕ, we have $\sin^2\phi = \dfrac{\sin^2\phi_0}{2}(1 + \cos 2d)$, and with the aid of trigonometric identities we can find expressions for $\sin^4\phi$ and $\sin^6\phi$, i.e.

$$\begin{aligned} \sin^2\phi &= \frac{\sin^2\phi_0}{2} (1 + \cos 2d), \\ \sin^4\phi &= \frac{\sin^4\phi_0}{8} (3 + 4 \cos 2d + \cos 4d), \\ \sin^6\phi &= \frac{\sin^6\phi_0}{32} (10 + 15 \cos 2d + 6 \cos 4d + \cos 6d). \end{aligned} \tag{92}$$

The values of $\sin^2\phi$, $\sin^4\phi$, $\sin^6\phi$ from (92) placed in (91) give (93)

$$S = a \left[\begin{aligned} & d - (e^2/4) \sin^2\phi_0 \int_{d_1}^{d_2} (1 - 3 \cos 2d)\, D\delta d \\ & - (3e^4/64) \sin^2\phi_0 \int_{d_1}^{d_2} \left[\begin{aligned} & (16 - 15 \sin^2\phi_0) + (16 - 20 \sin^2\phi_0) \cos 2d \\ & \qquad - 5 \sin^2\phi_0 \cos 4d \end{aligned} \right] D\delta d \\ & - (5e^6/512) \sin^4\phi_0 \int_{d_1}^{d_2} \left[\begin{aligned} & (72 - 70 \sin^2\phi_0) + (96 - 105 \sin^2\phi_0) \cos 2d \\ & + (24 - 42 \sin^2\phi_0) \cos 4d \\ & - 7 \sin^2\phi_0 \cos 6d \end{aligned} \right] D\delta d \end{aligned} \right]$$

Integration of (93) with respect to d leads to: (94)

$$S = a \left[\begin{aligned} & d - (e^2/4) \{ d [\sin^2\phi_0] - 3 \sin d [\sin^2\phi_0 \cos (d_1 + d_2)] \} \\ & - (3e^4/128) \left[\begin{aligned} & 32d [\sin^2\phi_0] - 30d [\sin^2\phi_0]^2 + 32 \sin d [\sin^2\phi_0 \cos (d_1 + d_2)] \\ & - 40 \sin d [\sin^2\phi_0] [\sin^2\phi_0 \cos (d_1 + d_2)] \\ & - 10 \sin 2d [\sin^2\phi_0 \cos (d_1 + d_2)]^2 + 5 \sin 2d [\sin^2\phi_0]^2 \end{aligned} \right] \\ & - (5e^6/1536) \left[\begin{aligned} & 216d [\sin^2\phi_0]^2 - 210d [\sin^2\phi_0]^3 + 288 \sin d [\sin^2\phi_0] [\sin^2\phi_0 \cos (d_1 + d_2)] \\ & - 315 \sin d [\sin^2\phi_0]^2 [\sin^2\phi_0 \cos (d_1 + d_2)] + 72 \sin 2d [\sin^2\phi_0 \cos (d_1 + d_2)]^2 \\ & - 126 \sin 2d [\sin^2\phi_0] [\sin^2\phi_0 \cos (d_1 + d_2)]^2 - 36 \sin 2d [\sin^2\phi_0]^2 \\ & + 63 \sin 2d [\sin^2\phi_0]^3 - 28 \sin 3d [\sin^2\phi_0 \cos (d_1 + d_2)]^3 \\ & + 21 \sin 3d [\sin^2\phi_0]^2 [\sin^2\phi_0 \cos (d_1 + d_2)] \end{aligned} \right] \end{aligned} \right]$$

From (79), with θ replaced by ϕ, we have

$$X = \frac{(\sin\phi_1 + \sin\phi_2)^2}{1+\cos d} + \frac{(\sin\phi_1 - \sin\phi_2)^2}{1-\cos d} = 2[\sin^2\phi_0], \tag{95}$$

$$Y = \frac{(\sin\phi_0 + \sin\phi_2)^2}{1+\cos d} - \frac{(\sin\phi_1 - \sin\phi_2)^2}{1-\cos d} = 2[\sin^2\phi_0 \cos(d_1 + d_2)].$$

Substituting from (95) in (94) we obtain finally

$$S = a\left[\begin{array}{l} d - (e^2/8)(Xd - 3Y\sin d) \\ -(3e^4/512)\left[\begin{array}{l} 64(Xd + Y\sin d) + (5\sin 2d - 30d)X^2 \\ \qquad - 40(\sin d)XY - 10(\sin 2d)Y^2 \end{array}\right] \\ -(5e^6/12{,}288)\left[\begin{array}{l} (432d - 72\sin 2d)X^2 + 576(\sin d)XY - 144(\sin 2d)Y^2 \\ + (63\sin 2d - 210\,d)X^3 + (21\sin 3d - 315\sin d)X^2Y \\ - 126(\sin 2d)XY^2 - 28(\sin 3d)Y^3 \end{array}\right] \end{array}\right] \tag{96}$$

If, in (96), we place $e^2 = 2f$, ignoring all terms above first order in f, one obtains the second of equations (80), or the Andoyer-Lambert approximation in terms of geodetic latitude, ϕ.

Now the Andoyer-Lambert forms can be obtained from other modifications of differential equations. For instance if the differential for arc length along the geodesic is taken in the form, [8] page 64,

$$ds = (N^2\cos^2\phi/N_0\cos\phi_0)\,d\lambda,\ N = a/(1 - e^2\sin^2\phi)^{1/2}; \tag{97}$$

if the differential of arc length from (84), after converting to geodetic latitude is written

$$ds = [(1 - e^2\sin^2\phi)^{-1/2} - (1 - e^2\sin^2\phi_0)^{-1/2}]\,D\delta d; \tag{98}$$

and if (97) and (98) are combined with the relationship $d\lambda = = (\sin a_c/\cos\phi)\,D\delta d = (\cos\phi_0/\cos^2\phi)\,D\delta d$ from the differential right triangles above with θ replaced by ϕ, one can write

$$(ds/a) = D\delta d + \left[\begin{array}{l}(1 - e^2\sin^2\phi)^{-1}(1 - e^2\sin^2\phi_0)^{1/2} - 1 \\ + (1 - e^2)^{1/2}\{(1 - e^2\sin^2\phi)^{-1/2} - (1 - e^2\sin^2\phi_0)^{-1/2}\}\end{array}\right] D\delta d \tag{99}$$

Expanding the expressions in (99) to first order terms in f, $e^2 = 2f$, equation (99) can be written in the integral form

$$S = a\left[d - f\int_{d_1}^{d_2}(2\sin^2\phi_0 - 3\sin^2\phi)\,D\delta d\right]. \tag{100}$$

Comparison of equations (100) and (91) (with $e^2 = 2f$) shows that (100) will again give the second of equations (80) or the Andoyer-Lambert Approximation in terms of geodetic latitude.

DERIVATIONS FROM EXPANSIONS OF FORSYTH

In reviewing the literature on geodetic computation one finds that A. R. Forsyth, [18], as early as 1895 had given some series expansions for geodetic arc length in terms of the flattening and certain spherical and elliptic parameters. On page 120 of his treatise one finds the expression

$$S_{12}/a = \nu_2' - \nu_1' - \tfrac{1}{4} c(\nu_2' - \nu_1') + (1/8) c(\sin 2\nu_2' - \sin 2\nu_1') \; . \qquad (101)$$

Now the correspondences between the parameters as used by Forsyth in deriving (101) and those used above in this investigation are to first order in f:

$$\nu_2' = d_2,\; \nu_1' = d_1,\; \nu_2' - \nu_1' = d_2 - d_1 = d,\;\; c = 2f \sin^2 \theta_0,$$

$$\sin 2\nu_2' - \sin 2\nu_1' = \sin 2d_2 - \sin 2d_1 = 2 \sin (d_2 - d_1) \cos (d_1 + d_2) = 2 \sin d \cos (d_1 + d_2)$$

so that equation (101) becomes equivalently

$$S = a\Big[d - (f/2)\, \{d[\sin^2\theta_0] - \sin d\, [\sin^2\theta_0 \cos (d_1 + d_2)]\}\Big],$$

which in turn by means of relations (79) can be written $S = a[d - (f/4)(Xd - Y \sin d)]$, and identified as the first Andoyer-Lambert form of equations (80).

On page 116 of Forsyth's treatise one finds the expression

$$\begin{aligned} S_{12}/a = {} & \nu_2 - \nu_1 + \xi\,\{(3/4) \cos^2 \alpha_0 (\sin 2\nu_2 - \sin 2\nu_1) - (\tfrac{1}{2})(\nu_2 - \nu_1) \cos^2 \alpha_0\} \\ & + \xi^2 \left[\begin{aligned} & (\tfrac{1}{2})(\nu_2 - \nu_1)^2 \cos^2 \alpha_0 \sin^3 \alpha_0 \sin \phi_1' \sin \phi_2'/\sin 2\phi_0 \\ & \quad + (\nu_2 - \nu_1)\,[(1/16) \cos^4 \alpha_0 + \cos^2 \alpha_0 \sin^2 \alpha_0] \\ {*2} \quad & + (3/8) \sin^3 \alpha_0 \cos^2 \alpha_0 (\sin 2\phi_2' - \sin 2\phi_1') \\ & - (3/4) \cos^2 \alpha_0 \sin^2 \alpha_0 (\sin 2\nu_2 - \sin 2\nu_1) \\ {*1} \quad & + (23/64) \cos^4 \alpha_0 (\sin 4\nu_2 - \sin 4\nu_1) \end{aligned}\right] \end{aligned} \qquad (102)$$

Now the equivalent relationships between Forsyth's parameters as used in (102) and the ones used in this investigation are:

$$\begin{aligned} & \nu_1 = d_1,\; \nu_2 = d_2,\; \nu_2 - \nu_1 = d_2 - d_1 = d,\; \xi = f,\; l_1 = \phi_1,\; l_2 = \phi_2, \\ & 2\phi_0 = \phi_2' - \phi_1' = \phi_2 - \phi_1 = \lambda_2 - \lambda_1 = \Delta\lambda,\; \cos \phi_1' = \cot \phi_0 \tan \phi_1 = \cos \phi_0 \cos d_1 \sec \phi_1 \\ & \sin \phi_1' = \sin d_1 \sec \phi_1,\; \cos \phi_2' = \cot \phi_0 \tan \phi_2 = \cos \phi_0 \cos d_2 \sec \phi_2 \\ & \sin \phi_2' = \sin d_2 \sec \phi_2,\; \cos \nu_1 = \cos d_1 = \sin \phi_1/\sin \phi_0\,, \\ & \cos \nu_2 = \cos d_2 = \sin \phi_2/\sin \phi_0,\; \alpha_0 = \frac{\pi}{2} - \phi_0, \text{ the relationship } \sin \alpha_0 \sin (\nu_2 - \nu_1) \\ & = \cos l_1 \cos l_2 \sin 2\phi_0 \text{ given on pages 106, 121 of Forsyth, [18],} \end{aligned} \qquad (103)$$

becomes $\cos \phi_0 \sin d = \cos \phi_1 \cos \phi_2 \sin \Delta\lambda$ in the notation of this investigation.

Assurance that Forsyth's α_0 is the complement of the geodetic latitude, ϕ_0, of the great elliptic arc is found from his expression, [18] page 106, which is

$$\tan \alpha_0 = \sin 2\phi_0 / \{(\tan l_1 + \tan l_2)^2 - 4 \tan l_1 \tan l_2 \cos^2\phi_0\}^{1/2}.$$

With equivalent substitutions from (103) and some trigonometric identities it will transform into

$$\tan \phi_0 = (\tan^2\phi_1 + \tan^2\phi_2 - 2 \tan \phi_1 \tan \phi_2 \cos \Delta\lambda)^{1/2} / \sin \Delta\lambda$$

which defines the vertex of the great elliptic arc. See equations (21) of this investigation.

A cursory check of the equations just preceding (102) in Forsyth's treatise revealed that the numerical coefficient of the second order term *1 in (102) should be 15/64 instead of 23/64. Then by use of relations (103) and (95) it was found that (102) could be written as

$$S = a \begin{bmatrix} d - (f/4)(Xd - 3Y \sin d) \\ \quad + (f^2/128)(AX - BY - CX^2 + DY^2 + EXY + FX^2Y + GX^3) \end{bmatrix} \tag{104}$$

where $A = 64d + 16d^2 \cot d$, $B = 96 \sin d + 16 d^2 \csc d - 48 \sin^2\Delta\lambda \csc d$, $C = 30d + 15 \sin 2d + 8d^2 \cot d + 12 \sin^2\Delta\lambda \cot d$, $D = 30 \sin 2d$, $E = 48 \sin d + 8d^2 \csc d - 36 \sin^2\Delta\lambda \csc d$, $F = 6 \sin^2\Delta\lambda \csc d$, $G = 6 \sin^2\Delta\lambda \cot d$.

Note that the first two terms of (104) are exactly the Andoyer-Lambert form given by the second of equations (80). But we apparently also have the second order term in the flattening. Thus, Forsyth had both so-called Andoyer-Lambert approximation forms as early as 1895 but they had not been recognized as such.

Equation (104) was used to compute several lines of known lengths. On those in which the term *2 of (102) was small, an improvement would be obtained by including the second order terms. On others, the error introduced would outweigh the first order correction, which could mean, since equation (104) is a power series in f, that the coefficient of the second order term in f is erroneous. Now examination of the second order terms of equations (82) and (96) shows no cubic terms in X and Y as are found in the second order term of (104). Hence Forsyth's paper [18], was reworked from the beginning and it was found that indeed the term *2 in (102) actually vanishes and reaffirmation was also made that the numerical coefficient of the term *1 of (102) should be 15/64 rather than 23/64. These errors are the result of carrying throughout the derivation the numerical factor 9/32 in the last term of the expression for δ, [18], section 17, page 98, when it should be 3/32. This affects the approximation equation for tan Φ, section 22, page 104. In the last term, the factor $-7 \sin^2\alpha$ should be $+5 \sin^2\alpha$. This continues to be reflected through section 27, pages 111 to 115, until the term is actually seen to vanish in collecting the terms together on page 115. Also on page 115, omission of a factor ½ in use of a trigonometric identity in the third line from the bottom gave the printed value ¼ for the numerical coefficient of

$\cos^4 \alpha_0 \sin 4\nu$ when it should be 1/8. This leads in turn to the printed value 23/64 as given on page 116 when it should be 15/64.

After the two errors in Forsyth's second order term in f had been detected, two papers were found which are concerned with the Forsyth derivation, Wassef 1948, [19], and Gougenheim 1950, [20]. Wassef purports to give the corrected version of Forsyth's second order term but he includes the term *2 in (102) and he gives 15/23 for the numerical coefficient of *1 in (102). Hence Wassef's results are erroneous and useless. Gougenheim, unaware of Forsyth's work, had developed his formulae independently and he has the term *2 in (102) missing in his derivation and the correct numerical coefficient 15/64 for *1 of (102). His formula for the second order term is (in the notation of Forsyth)

$$+\xi^2 \left[\begin{aligned} &-(1/2) \frac{(\nu_2-\nu_1)^2}{\cot\nu_2 - \cot\nu_1} \cos^2\alpha_0 \sin^2\alpha_0 + (1/16) \quad (\nu_2-\nu_1)(\cos^2\alpha_0 + 15\cos^2\alpha_0\sin^2\alpha_0) \\ &- (3/4)\cos^2\alpha_0 \sin^2\alpha_0 (\sin 2\nu_2 - \sin 2\nu_1) \\ &+ (15/64)\cos^4\alpha_0 (\sin 4\nu_2 - \sin 4\nu_1) \end{aligned} \right] \tag{105}$$

Since the last two terms of (105) are the same as the last two of (102), as corrected, we have only to show that

$$\begin{aligned} &(1/16)\cos^4\alpha_0 + \cos^2\alpha_0 \sin^2\alpha_0 \equiv (1/16)(\cos^2\alpha_0 + 15\cos^2\alpha_0\sin^2\alpha_0), \\ &1/(\cot\nu_1 - \cot\nu_2) \equiv (\sin\alpha_0 \sin\phi_1' \sin\phi_2')/\sin 2\phi_0. \end{aligned} \tag{106}$$

Writing the right member of the first of (106) as

$$\begin{aligned} &(1/16)\cos^2\alpha_0 + (15/16)\cos^2\alpha_0\sin^2\alpha_0 + (1/16)\cos^4\alpha_0 - (1/16)\cos^2\alpha_0(1-\sin^2\alpha_0) \\ &\equiv (1/16)\cos^4\alpha_0 + (1/16)\cos^2\alpha_0 + (15/16)\cos^2\alpha_0\sin^2\alpha_0 \\ &\qquad\qquad - (1/16)\cos^2\alpha_0 + (1/16)\cos^2\alpha_0\sin^2\alpha_0 \\ &\equiv (1/16)\cos^4\alpha_0 + \cos^2\alpha_0\sin^2\alpha_0. \end{aligned}$$

From relations (103) we have

$$\sin\alpha_0 \sin(\nu_2-\nu_1) = \cos l_1 \cos l_2 \sin 2\phi_0 \quad \text{or}$$

$$\frac{\sin\alpha_0}{\sin 2\phi_0} = \frac{\cos l_1 \cos l_2}{\sin(\nu_2-\nu_1)} \tag{107}$$

$$\frac{\sin\alpha_0 \sin\phi_1' \sin\phi_2'}{\sin 2\phi_0} = \frac{\cos l_1 \sin\phi_1' \cdot \cos l_2 \sin\phi_2'}{\sin\nu_2\cos\nu_1 - \cos\nu_2\sin\nu_1} = \frac{\dfrac{\cos l_1 \sin\phi_1'}{\sin\nu_1} \cdot \dfrac{\cos l_2\sin\phi_2'}{\sin\nu_2}}{\cot\nu_1 - \cot\nu_2}$$

From pages 111, 117 of Forsyth find:

$\tan \phi_1' \sin \alpha_0 = \tan \nu_1$, $\cos \phi_1' = \tan \alpha_0 \tan l_1$, $\cos \nu_1 \cos \alpha_0 = \sin l_1$,

$\tan \phi_2' \sin \alpha_0 = \tan \nu_2$, $\cos \phi_2' = \tan \alpha_0 \tan l_2$, $\cos \nu_2 \cos \alpha_0 = \sin l_2$,

whence

$$\frac{\cos l_1 \sin \phi_1'}{\sin \nu_1} = \frac{\sin l_1}{\cos \nu_1 \cos \alpha_0} = 1, \tag{108}$$

$$\frac{\cos l_2 \sin \phi_2'}{\sin \nu_2} = \frac{\sin l_2}{\cos \nu_2 \cos \alpha_0} = 1.$$

The values from (108) placed in (107) prove the second of (106) and thus Gougenheim's paper provides an independent check of the corrections given here to Forsyth's second order term. Gougenheim also gave formulae for azimuths, convergence of the meridians, and difference in longitude between the spheroidal and spherical (elliptical) vertices of geodesics in terms of the same variables. The importance of Gougenheim's work has not been recognized. He has had a correct expansion including the second order term in the flattening, in print since 1950.

THE FORSYTH-ANDOYER-LAMBERT TYPE APPROXIMATION IN GEODETIC LATITUDE WITH SECOND ORDER TERMS

With the corrections to (102), i.e. with the numerical coefficient of *1 as 15/64 and the term *2 omitted, equation (102) may be written, with relations (103) and (95), as

$$S = a[d - (f/4)(Xd - 3Y \sin d) + (f^2/128)(AX + BY + CX^2 + DXY + EY^2)], \tag{109}$$

where a, f are the semimajor axis and flattening of the reference ellipsoid; d is the spherical distance between the points $P_1(\phi_1, \lambda_1,)$, $P_2(\phi_2, \lambda_2)$ on the ellipsoid given by some spherical formula as $\cos d = \sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2 \cos \Delta\lambda$; ϕ is geodetic latitude, λ is longitude, $\Delta\lambda = \lambda_2 - \lambda_1$; $A = 64d + 16d^2 \cot d$, $D = 48 \sin d + 8d^2 \csc d$, $B = -2D$, $E = 30 \sin 2d$,

$$C = -(30d + 8d^2 \cot d + E/2),\ X = \frac{(\sin \phi_1 + \sin \phi_2)^2}{1 + \cos d} + \frac{(\sin \phi_1 - \sin \phi_2)^2}{1 - \cos d},$$

$$Y = \frac{(\sin \phi_1 + \sin \phi_2)^2}{1 + \cos d} - \frac{(\sin \phi_1 - \sin \phi_2)^2}{1 - \cos d};\ d = d_2 - d_1,$$ where d_1 and d_2 are spherical distances from the vertex of the great elliptic arc to the points $P_1(\phi_1, \lambda_1)$, $P_2(\phi_2, \lambda_2)$.

Now by factoring sin d out of every term of (109) and using the azimuth formulae as given by Lambert, [13], we can, by means of trigonometric identities, arrange equations (109) in a form more convenient for computing as follows:

Given on the reference ellipsoid the points P_1 (ϕ_1, λ_1), P_2 (ϕ_2, λ_2), ϕ is geodetic latitude, λ is longitude, P_2 is west of P_1 with west longitudes considered positive.

With $\phi_m = (1/2)(\phi_1 + \phi_2)$, $\Delta\phi_m = (1/2)(\phi_2 - \phi_1)$, $\Delta\lambda = \lambda_2 - \lambda_1$, $\Delta\lambda_m = (1/2)\Delta\lambda$;

Let: $k = \sin\phi_m \cos\Delta\phi_m$, $K = \sin\Delta\phi_m \cos\phi_m$,

$$H = \cos^2\Delta\phi_m - \sin^2\phi_m = \cos^2\phi_m - \sin^2\Delta\phi_m,$$

$$L = \sin^2\Delta\phi_m + H\sin^2\Delta\lambda_m = \sin^2(d/2),\ 1 - L = \cos^2(d/2),\ \cos d = 1 - 2L,\ t = \sin^2 d = 4L(1-L),$$

$$U = 2k^2/(1-L),\ V = 2K^2/L,\ X = U + V,\ Y = U - V,$$

$$T = d/\sin d = 1 + (t/6) + 3(t^2/40) + 5(t^3/112) + 35(t^4/1152) + 63(t^5/2816) + \text{-----},$$

$$E = 30\cos d,\ A = 4T(8 + TE/15),\ D = 4(6 + T^2),\ B = -2D,\ C = T - \tfrac{1}{2}(A + E), \tag{110}$$

$$S = a\sin d\,[T - (f/4)(TX - 3Y) + (f^2/64)\{X(A + CX) + Y(B + EY) + DXY\}];$$

$$\sin(\alpha_2 + \alpha_2) = (K\sin\Delta\lambda)/L,\ \sin(\alpha_2 - \alpha_1) = (k\sin\Delta\lambda)/(1 - L)$$

$$(\tfrac{1}{2})(\delta\alpha_2 + \delta\alpha_1) = -(f/2)H(T+1)\sin(\alpha_2 + \alpha_1),\ (\tfrac{1}{2})(\delta\alpha_2 - \delta\alpha_1) = -(f/2)H(T-1)\sin(\alpha_2 - \alpha_1),$$

$$\alpha_{1-2} = \alpha_1 + \delta\alpha_1,\ \alpha_{2-1} = \alpha_2 + \delta\alpha_2.$$

Note that the quantities H, T, L, k, K enter into both distance and azimuth formulas.

Figure (21) shows an arrangement of equations (110) for desk computing using an ordinary ten bank electriç desk calculator and Peters eight place tables of trigonometric functions. It is arranged to show the contribution of both the first and second order terms in the flattening.

Table 4 summarizes the results of computations over 17 lines of known lengths and azimuths. The computations are given in Appendix 3. Part of these lines were used in the computations of Appendix 2. The first 11 lines are from two ACIC publications [12], lines 12 through 17 are Coast and Geodetic Survey specially computed lines, [22].

Note that all distances are within one meter and azimuths are within one second which was the objective since this is adequate for any operational requirement. Other advantages are (1) no conversion to parametric latitudes, (2) no square root calculation, (3) for desk computers the only tabular data required is a table of the natural trigonometric functions as Peters eight place tables, (4) the formulas are adaptable to high speed computers, (5) about the same accuracy is obtained over all lines in all azimuths and latitudes.

EXPANSION TO SECOND ORDER TERMS IN f USING PARAMETRIC LATITUDE

Forsyth [18], gave an expansion of the geodesic to first order in the elliptic modulus $c = (e^2\cos^2\alpha)/(1 - e^2\sin^2\alpha)$ where α is the complement of the parametric latitude of the vertex of the geodesic. (See pages 118–120 of his treatise). We will follow the Forsyth method and

DISTANCE COMPUTING FORM, FORSYTH-ANDOYER-LAMBERT
TYPE APPROXIMATION WITH SECOND ORDER TERMS
(No conversion to parametric latitudes)
Clarke Spheroid 1866, a = 6,378,206.4 meters
$f/2 = 0.00169503765$, $f/4 = 0.000847518825$, $f^2/64 = 0.1795720390 \times 10^{-6}$
1 radian = 206,264.8062 seconds

ϕ_1 8° 58′ 25.0″ — 1. PANAMA — λ_1 79° 34′ 24.0″

ϕ_2 21 26 06.0 — 2. HAWAII — λ_2 158 01 33.0

$\phi_m = \frac{1}{2}(\phi_1 + \phi_2)$ 15° 12′ 15.″5 — 2. Always west of 1. — $\Delta\lambda = \lambda_2 - \lambda_1$ 78 27 09.0

$\Delta\phi_m = \frac{1}{2}(\phi_2 - \phi_1)$ 6 13 50.5 — $\Delta\lambda_m = \frac{1}{2}\Delta\lambda$ 39 13 34.5

$\sin \phi_m$ +.26226170 — $\sin \Delta\phi_m$ +.10853193 — $\sin \Delta\lambda$ +.97975909

$\cos \phi_m$ +.96499679 — $\cos \Delta\phi_m$ +.99409297 — $\sin \Delta\lambda_m$ +.63238428

$k = \sin \phi_m \cos \Delta\phi_m$ +.260712512 — $K = \sin \Delta\phi_m \cos \phi_m$ +.104732963

$H = \cos^2 \Delta\phi_m - \sin^2 \phi_m = \cos^2\phi_m - \sin^2\Delta\phi_m$ +.919439630 — $1 - L$ +.62052783

$L = \sin^2\Delta\phi_m + H \sin^2\Delta\lambda_m$ +.37947217 — $\cos d = 1 - 2L$ +.24105566

$d +$ 1.327342885 — $\sin d +$ .97051129 — $T = d/\sin d +$ 1.367673822

$U = 2k^2/(1-L)$ +.219074828 — $V = 2K^2/L$ +.0578118469 — $E = 30 \cos d$ +7.2316698

$X = U + V$ +.276886675 — $Y = U - V$ +.161262981 — $D = 4(6 + T^2)$ +31.48212675

$A = 4T(8 + ET/15)$ +47.3727803 — $C = T - \frac{1}{2}(A+E)$ −25.93455125 — $B = -2D$ −62.9642535

$X(A + CX)$ +11.128587321 — $Y(B + EY)$ −9.96573823 — DXY +1.405726406

$(TX - 3Y)$ −.105093286 — $\delta f = -(f/4)(TX - 3Y)$ $+8.90728 \times 10^{-5}$

$T + \delta f$ +1.36776290 — $S_1 = a \sin d\,(T + \delta f)$ 8,466,618.26 meters

$\Sigma = X(A + CX) + Y(B + EY) + DXY$ +2.5685755 — $\delta f^2 = +(f^2/64)\Sigma$ $+4.6124 \times 10^{-7}$

$T + \delta f + \delta f^2$ +1.36776336 — $S_2 = a \sin d\,(T + \delta f + \delta f^2)$ 8,466,621.11 meters

$\sin(\alpha_2 + \alpha_1) = (K \sin \Delta\lambda)/L$ +.27041001 — $\alpha_2 + \alpha_1$ 375° 41′ 19.″197

$\sin(\alpha_2 - \alpha_1) = (k \sin \Delta\lambda)/(1-L)$ +.41164222 — $\alpha_2 - \alpha_1$ 155 41 31.161

$\frac{1}{2}(\delta\alpha_1 + \delta\alpha_2) = -(f/2) H (T + 1) \sin(\alpha_2 + \alpha_1)$ $-9.97808513 \times 10^{-4}$ — $\delta\alpha_1$ $-.761931734 \times 10^{-3}$

$\frac{1}{2}(\delta\alpha_2 - \delta\alpha_1) = -(f/2) H (T - 1) \sin(\alpha_2 - \alpha_1)$ $-2.35876779 \times 10^{-4}$ — $\delta\alpha_2$ $-1.233685292 \times 10^{-3}$

α_1 109° 59′ 54.″018 — α_2 265° 41′ 25.″179

$\delta\alpha_1$ — 2 37.160 — $\delta\alpha_2$ — 4 14.466

α_{1-2} 109 57 16.858 — α_{2-1} 265 37 10.713

$\alpha_{1-2} = \alpha_1 + \delta\alpha_1$ — $\alpha_{2-1} = \alpha_2 + \delta\alpha_2$

Figure 21.

TABLE 4

Summary of Computations

No.	Approx. Lat. °	Approx. Az. °	True Length S(Meters)	Computed Length $S_1(\delta f)$ Meters	Computed Length $S_2(\delta f^2)$ Meters	Computed Length $S_1 - S$ Meters	Computed Length $S_2 - S$ Meters	True Azimuths ° ' "	Computed Azimuths ° ' "
								45 26 01.69	00.44
1	40	45	80,466.49	67.25	67.02	+ 0.76	+ 0.53	224 59 59.997	58.76
								90 15 17.48	17.51
2	10	90	160,932.96	32.99	32.96	+ 0.03	0.0	270 0 0	00.02
								97 52 01.06	01.11
3	70	90	321,865.91	62.98	65.64	− 2.93	− 0.27	269 59 59.95	270 00 00.03
								45 37 46.11	44.97
4	10	45	482,798.87	94.74	99.23	− 4.13	+ 0.36	224 59 59.73	58.63
								58 50 31.60	31.30
5	70	45	643,732.43	27.96	32.44	− 4.47	+ 0.01	225 00 00.15	224 59 59.86
								91 16 14.93	14.87
6	10	90	804,664.78	65.22	65.10	+ 0.44	+ 0.32	270 0 0	269 59 59.98
								49 52 15.53	14.35
7	40	45	804,664.77	66.62	64.75	+ 1.95	− 0.02	224 59 59.99	58.83
								89 55 22.83	22.64
8	70	45	1,609,329.06	15.61	29.04	−13.45	− 0.02	224 59 59.96	59.83
								119 54 41.26	41.40
9	40	90	4,827,984.25	83.17	85.09	− 1.08	+ 0.84	270 00 00.12	269 59 59.61
								138 23 42.76	42.39
10	40	45	9,655,969.75	72.49	70.13	+ 2.74	+ 0.38	225 00 00.28	00.67
								159 54 37.21	37.78
11	70	90	9,655,977.15	63.63	77.01	−13.52	− 0.14	270 00 00.02	00.81
				599,	600,			260 17 09.79	09.78
12	70	95	600,000.00	995.26	000.24	− 4.74	+ 0.24	95 0 0	94 59 59.93
				900,	900,			50 0 0	49 59 59.20
13	60	50	900,000.00	000.56	000.23	+ 0.56	+ 0.23	221 03 33.54	32.73
								128 33 08.34	09.17
14	25	50	979,251.25	247.67	251.45	− 3.58	+ 0.20	305 38 13.25	14.18
								35 16 34.25	33.34
15	60	35	1,232,647.21	652.17	647.21	+ 4.96	0.0	207 08 33.82	32.91
								109 57 17.41	16.86
16	20	70	8,466,621.01	618.26	621.11	− 2.75	+ 0.10	265 37 10.59	10.71
								15 48 17.67	16.94
17	55	15	10,102,069.06	057.93	069.86	−11.13	+ 0.80	190 39 32.21	31.45

extend the results to second order in c and subsequently to second order in f since c can be expressed as a series in f.

The quantities needed to achieve the approximation are found in or derived from the results of Forsyth's work, pages 86, 97–105. We list them here for reference in the development.

$$\Phi = \phi + \frac{c}{2} u' \sec \alpha \tan \alpha \, [1 + \frac{c}{8}(1 - 6 \tan^2\alpha)] \qquad \text{111a}$$

$$u' = \nu' + c\, U + c^2 V \qquad \text{111b}$$

$$\phi = \phi' + c\, \Omega + c^2 \Psi \qquad \text{111c}$$

$$\alpha = \alpha_0 + cA \cot \alpha_0 + c^2 B \qquad \text{111d}$$

$$\mathrm{cn}\, u = \cos u' \{ 1 - \tfrac{1}{4} c \sin^2 u' - \frac{c^2}{64} \sin^2 u' (7 + 4 \cos^2 u') \} \qquad \text{111e}$$

$$c = (e^2 \cos^2\alpha)/(1 - e^2 \sin^2\alpha), \; e^2 = 2f - f^2, \; e^4 = 4f^2$$

$$c = 2f \cos^2\alpha + f^2 \cos^2\alpha \, (3 - 4 \cos^2\alpha) \qquad \text{111f}$$

$$\cos \theta = \mathrm{cn}\, u \cos \alpha \qquad \text{111g}$$

$$\tan \Phi = \tan u' \csc \alpha \, [1 + \tfrac{1}{4}c + (1/64) c^2 (9 - 2 \sin^2\nu' - 4 \tan^2 \alpha_0)] \qquad \text{111h}$$

$$\frac{s}{a} = (1 - e^2 \sin^2\alpha)^{1/2} E(u)$$

$$= u' + \frac{c}{4}[\sin 2u' - (1 + 2 \tan^2\alpha)\, u'] \qquad \text{111i}$$

$$+ \frac{c^2}{64} [\sin 4u' + 4 \sin 2u' (1 - 2 \tan^2\alpha) + \{ 8 \tan^2\alpha \, (1 + 3 \tan^2\alpha) - 3 \}\, u']$$

$$\sin \alpha = \sin \alpha_0 [1 + c A \cot^2\alpha_0 + c^2 \cot \alpha_0 \, (B - \tfrac{1}{2} A^2 \cot \alpha_0)] \qquad \text{111j}$$

$$\cos \alpha = \cos \alpha_0 \, [1 - c\, A - c^2 \tan \alpha_0 \, (B + \tfrac{1}{2} A^2 \cot^3\alpha_0)] \qquad \text{111k}$$

$$\tan \alpha = \tan \alpha_0 \, [1 + c\, A \csc^2\alpha_0 + c^2 \csc^2\alpha_0 \, (A^2 + B \tan \alpha_0)] \qquad \text{111m}$$

$$\sec \alpha = \sec \alpha_0 \, [1 + c\, A + c^2 \tan \alpha_0 \, (B + A^2 \cot \alpha_0 \, \{ 1 + \tfrac{1}{2} \cot^2\alpha_0 \})] \qquad \text{111n}$$

$$\csc \alpha = \csc \alpha_0 \, [1 - c\, A \cot^2 \alpha_0 - c^2 \cot \alpha_0 \, \{ B - \tfrac{1}{2} A^2 \cot \alpha_0 \, (1 + 2 \cot^2 \alpha_0) \}] \qquad \text{111o}$$

$$\sin u' = \sin \nu' [1 + c\, U \cot \nu' + c^2 (V \cot \nu' - U^2/2)] \qquad \text{111p}$$

$$\cos u' = \cos \nu' [1 - c\, U \tan \nu' - c^2 \, (V \tan \nu' + U^2/2)] \qquad \text{111q}$$

$$\tan u' = \tan \nu' + c\, U \sec^2 \nu' + c^2 \sec^2 \nu' (V + U^2 \tan \nu') \qquad \text{111r}$$

$$\sin 2u' = \sin 2\nu' (1 + 2c\, U \cot 2\nu') \text{ (to first order in c)}$$

$$\tan \phi' = \tan \nu' \csc \alpha_0 , \; 1 + \tan^2 \nu' \csc^2\alpha_0 = \sec^2 \phi' \qquad \text{111s}$$

$$U = - (A \cot \nu' + (1/8) \sin 2\nu'), \; A = - (\nu'/2) \tan^2\alpha_0 \tan \nu'$$

$$\Omega + (\nu'/2) \sin \alpha_0 \sec^2 \alpha_0 = - A \csc^2 \alpha_0 \cot \phi' \qquad \text{111t}$$

In these formulas, α_0 is the complement of the parametric latitude of the vertex of the great elliptic arc. To see this, find on page 119 of Forsyth, the expression

$$\sin \alpha_0 = (\tan \phi_0)/[(p \sec^2\phi_0 - 1)(p' \sec^2\phi_0 + 1)]^{1/2},$$

$$\text{where } \quad p = \sin^2 \tfrac{1}{2}(\theta_1 + \theta_2)/\sin \theta_1 \sin \theta_2 \tag{112}$$

$$p' = \cos^2 \tfrac{1}{2}(\theta_1 + \theta_2)/\sin \theta_1 \sin \theta_2$$

Now replace Forsyth's θ_1 and θ_2 by $90 - \theta_1$, $90 - \theta_2$ respectively and his ϕ_0 by $\Delta\lambda/2$. Then find:

$$\tan \phi_0 = \tan (\Delta\lambda/2) = (1 - \cos \Delta\lambda)/\sin \Delta\lambda$$

$$p \sec^2\phi_0 - 1 = [(1 - \cos \Delta\lambda)/\sin^2\Delta\lambda]\,(1 + \sec \theta_1 \sec \theta_2 - \tan \theta_1 \tan \theta_2) - 1 \tag{113}$$

$$p' \sec^2\phi_0 + 1 = [(1 - \cos \Delta\lambda)/\sin^2\Delta\lambda]\,(-1 + \sec \theta_1 \sec \theta_2 + \tan \theta_1 \tan \theta_2) + 1$$

The values from (113) placed in (112) give

$$\sin \alpha_0 = \sin \Delta\lambda/(\tan^2\theta_1 + \tan^2\theta_2 - 2 \tan \theta_1 \tan \theta_2 \cos \Delta\lambda + \sin^2\Delta\lambda)^{1/2} \tag{114}$$

Now the right member of (114) is $\cos \theta_0$ where θ_0 is the parametric latitude of the vertex of the great elliptic arc [17] . (See also GEODESICS AND PLANE ARCS ON AN OBLATE SPHEROID, L. E. Ward, American Mathematical Monthly, Aug.—Sept., 1943 page 427).

From 111a, 111b, 111c, 111m, 111n we have, retaining terms to c^2 inclusive:

$$\Phi = \phi' + c\left(\Omega + \frac{\nu'}{2} \sec \alpha_0 \tan \alpha_0\right) \tag{115}$$

$$+ c^2 [\Psi + \tfrac{1}{2} \sec \alpha_0 \tan \alpha_0 \{U + A\nu'(1 + \csc^2\alpha_0) + (1/8)\,\nu'(1 - 6 \tan^2 \alpha_0)\}]$$

If R, S are the coefficients respectively of c and c^2 in (115), then

$$\tan \Phi = \tan \phi' + c \sec^2\phi' R + c^2 \sec^2\phi' (S + R^2 \tan \phi') \tag{116}$$

With the values of R and S from (115) and the values of $\Omega + (\nu'/2) \sec \alpha_0 \tan \alpha_0$ and U from 111t, cot ϕ' from 111s, we can write (116) as

$$\tan \Phi = \tan \phi' - c A \cot \nu' \csc \alpha_0 \sec^2\phi' \tag{117}$$

$$+ c^2 \sec^2\phi' \left[\begin{array}{l} \Psi + A^2 \cot \nu' \csc^3 \alpha_0 \\ + \tfrac{1}{2} \sin \alpha_0 \sec^2\alpha_0 \left[\begin{array}{l} A[\nu'(1 + \csc^2\alpha_0) - \cot \nu'] \\ - (1/8) \sin 2\nu' + \dfrac{\nu'}{8}(1 - 6 \tan^2\alpha_0) \end{array}\right] \end{array}\right]$$

From 111h, 111o, 111r we write a second formula for $\tan\Phi$:

$$\tan\Phi = \tan\nu'\csc a_0 - cA\,(\csc^2\nu' + \cot^2 a_0)\tan\nu'\csc a_0$$
$$+ c^2\tan\nu'\csc a_0 \left[\begin{array}{l} V\sec\nu'\csc\nu' - B\cot a_0 + (9/64) + (1/32)\sin^2\nu' \\ + \dfrac{A}{4}(2-\csc^2\nu') - (1/16)\sec^2 a_0 \\ + A^2(\csc^2\nu'\csc^2 a_0 + \cot^4 a_0 + \tfrac{1}{2}\cot^2 a_0) \end{array}\right] \qquad (118)$$

From 111g, 111e, 111k, 111p, 111q, 111t we can write:

$$\cos\theta = \cos a_0\cos\nu' + c\cdot 0 \qquad (119)$$
$$+ c^2\cos a_0\cos\nu' \left(\begin{array}{l} \dfrac{A}{4}\cos 2\nu' - V\tan\nu' - (5/64)\sin^2\nu' - (3/32)\sin^4\nu' \\ - B\tan a_0 - A^2(1 + \tfrac{1}{2}\cot^2 a_0 + \tfrac{1}{2}\cot^2\nu') \end{array}\right)$$

Now in (119), the coefficient of c was zero as it should be and the coefficient of c^2 must be zero since $\cos\theta = \cos a_0\cos\nu'$. Placing the coefficient of c^2 in (119) equal to zero find:

$$-B\cot a_0 = A^2(1 + \tfrac{1}{2}\cot^2 a_0 + \tfrac{1}{2}\cot^2\nu')\cot^2 a_0 - \frac{A}{4}\cos 2\nu'\cot^2 a_0$$
$$+ V\tan\nu'\cot^2 a_0 + (5/64)\sin^2\nu'\cot^2 a_0 + (3/32)\sin^4\nu'\cot^2 a_0 \qquad (120)$$

With the value of $-B\cot a_0$ from (120) placed in the second order term of (118) and with some manipulation through the identities 111s, we can write (118) as:

$$\tan\Phi = \tan\nu'\csc a_0 - c\,A\cot\nu'\csc a_0\sec^2\phi'$$
$$+ c^2\csc a_0\sec^2\phi' \left(\begin{array}{l} A^2\cot\nu'(1 + (3/2)\cot^2 a_0) + V \\ + \dfrac{A}{4}(\sin 2\nu' - \cot\nu') + (1/16)\sin 2\nu' \\ - (3/256)\sin 4\nu' - (1/32)\sin 2\nu'\tan^2 a_0 \end{array}\right) \qquad (121)$$

From (117) and (121), since $\tan\phi' = \tan\nu'\csc a_0$ from 111s, the coefficients of the terms in c and c^2 must be respeceively equal. Equating the second order terms in (117) and (121) and solving for V we find:

$$V = \Psi\sin a_0 - \tfrac{1}{2}A^2\cot\nu'\cot^2 a_0 \qquad (122)$$
$$+ \frac{A}{4}\left[2\nu'\tan^2 a_0(1 + \csc^2 a_0) - \sin 2\nu' + \cot\nu'(1 - 2\tan^2 a_0)\right]$$
$$+ \frac{\nu'}{16}\tan^2 a\,(1 - 6\tan^2 a) - \frac{\sin 2\nu'}{16} + \frac{3\sin 4\nu'}{256} - \frac{\tan^2 a_0\sin 2\nu'}{32}$$

From 111i, 111b, 111m, 111p, 111q, the value of U in terms of A from 111t, and V from (122) we may write:

$$\frac{S}{a} = \nu' + c\left[(1/8)\sin 2\nu' - A\cot\nu' - \frac{\nu'}{4}(1 + 2\tan^2\alpha_0)\right] \tag{123}$$

$$+ c^2\left[\begin{array}{l}\Psi\sin\alpha_0 - \tfrac{1}{2}A^2\cot^2\alpha_0\cot\nu' + \dfrac{A}{4}(\sin 2\nu' - 2\nu') \\ + (1/256)\left[8\sin 2\nu'(1 - 3\tan^2\alpha_0) - \sin 4\nu'\right] + (3/64)\,\nu'(4\tan^2\alpha_0 - 1)\end{array}\right]$$

Referring (123) to the end points of the geodesic arc we have:

$$\frac{S}{a} = (\nu_2' - \nu_1') + c\left[(1/8)(\sin 2\nu_2' - \sin 2\nu_1') - A(\cot\nu_2' - \cot\nu_1') - \tfrac{1}{4}(\nu_2' - \nu_1')(1 + 2\tan^2\alpha_0)\right]$$

$$+ c^2\left[\begin{array}{l}-\tfrac{1}{2}A^2\cot^2\alpha_0(\cot\nu_2' - \cot\nu_1') + \dfrac{A}{4}\left[(\sin 2\nu_2' - \sin 2\nu_1') - 2(\nu_2' - \nu_1')\right] \\ + (1/256)\left[8(1 - 3\tan^2\alpha_0)(\sin 2\nu_2' - \sin 2\nu_1') - (\sin 4\nu_2' - \sin 4\nu_1')\right] \\ \quad + (3/64)(\nu_2' - \nu_1')(4\tan^2\alpha_0 - 1)\end{array}\right] \tag{124}$$

Note that the term $\Psi \sin \alpha_0$ vanishes in (124).

From 111t we have from the expression forA that:

$$-A(\cot\nu_2' - \cot\nu_1') = \frac{\tan^2\alpha_0}{2}(\nu_2' - \nu_1'), \tag{125}$$

$$A = \tfrac{1}{4}(\nu_2' - \nu_1')\tan^2\alpha_0\left[\cot(\nu_2' - \nu_1') - \csc(\nu_2' - \nu_1')\cos(\nu_1' + \nu_2')\right]$$

We list also for reference the identities:

$$\sin 2\nu_2' - \sin 2\nu_1' = 2\sin(\nu_2' - \nu_1')\cos(\nu_1' + \nu_2'), \tag{126}$$

$$\sin 4\nu_2' - \sin 4\nu_1' = 2\sin 2(\nu_2' - \nu_1')\left[2\cos^2(\nu_1' + \nu_2') - 1\right]$$

Applying (125) and (126) to (124) we obtain:

$$\frac{S}{a} = (\nu_2' - \nu_1') - (c/4)\left[(\nu_2' - \nu_1') - \sin(\nu_2' - \nu_1')\cos(\nu_1' + \nu_2')\right] \tag{127}$$

$$+ c^2\left[\begin{array}{l}\dfrac{A}{2}\sin(\nu_2' - \nu_1')\cos(\nu_1' + \nu_2') - \dfrac{A}{4}(\nu_2' - \nu_1') + (3/64)(\nu_2' - \nu_1')(4\tan^2\alpha_0 - 1) \\ + (1/16)(1 - 3\tan^2\alpha_0)\sin(\nu_2' - \nu_1')\cos(\nu_1' + \nu_2') \\ - (1/128)\sin 2(\nu_2' - \nu_1')\left[2\cos^2(\nu_1' + \nu_2') - 1\right]\end{array}\right]$$

Note that the first two terms of (127) are equivalent to Forsyth's equation, page 120 of his treatise.

Now for the value of c, we find on page 97 of Forsyth, that for approximations involving f^2 (second order in the flattening) a value of α that is accurate up to f inclusive must be substituted in the first term of c. Hence from 111d, 111f, 111k we have

$$c = 2f\cos^2\alpha_0 + 3f^2\cos^2\alpha_0 - 4f^2\cos^4\alpha_0(1 + 2A). \tag{128}$$

This value of c placed in (127) with the value of A from (125) gives:

$$\frac{S}{a} = (\nu_2' - \nu_1') - (f/2)\cos^2\alpha_0\,[(\nu_2' - \nu_1') - \sin(\nu_2' - \nu_1')\cos(\nu_1' + \nu_2')] \tag{129}$$

$$+ f^2 \begin{bmatrix} \tfrac{1}{4}(\nu_2' - \nu_1')^2 \cot(\nu_2' - \nu_1')\cos^2\alpha_0 - \tfrac{1}{4}(\nu_2' - \nu_1')^2 \cot(\nu_2' - \nu_1')\cos^4\alpha_0 \\ -\tfrac{1}{4}(\nu_2' - \nu_1')^2 \csc(\nu_2' - \nu_1')\cos^2\alpha_0 \cos(\nu_1' + \nu_2') \\ +\tfrac{1}{4}(\nu_2' - \nu_1')^2 \csc(\nu_2' - \nu_1')\cos^4\alpha_0 \cos(\nu_1' + \nu_2') \\ -(1/16)\sin 2(\nu_2' - \nu_1')\cos^4\alpha_0 \cos^2(\nu_1' + \nu_2') \\ +(1/16)(\nu_2' - \nu_1')\cos^4\alpha_0 + (1/32)\sin 2(\nu_2' - \nu_1')\cos^4\alpha_0 \end{bmatrix}$$

Now in (129) let $\alpha_0 = 90° - \theta_0$, $d_1 = \nu_1'$, $d_2 = \nu_2'$, $d = d_2 - d_1 = \nu_2' - \nu_1'$ and the equation becomes:

$$\frac{S}{a} = d - (f/2)\,[d \sin^2\theta_0 - \sin d \sin^2\theta_0 \cos(d_1 + d_2)] \tag{130}$$

$$+ f^2 \begin{bmatrix} \tfrac{1}{4} d^2 \cot d \sin^2\theta_0 - \tfrac{1}{4} d^2 \cot d \sin^4\theta_0 \\ - \tfrac{1}{4} d^2 \csc d \sin^2\theta_0 \cos(d_1 + d_2) \\ + \tfrac{1}{4} d^2 \csc d \sin^4\theta_0 \cos(d_1 + d_2) \\ - (1/16)\sin 2d \sin^4\theta_0 \cos^2(d_1 + d_2) + (1/16)\, d \sin^4\theta_0 + (1/32)\sin 2d \sin^4\theta_0 \end{bmatrix}$$

Since θ_0 is the parametric latitude of the vertex of the Great elliptic arc, we have (or may place)

$$X = \frac{(\sin\theta_1 + \sin\theta_2)^2}{1 + \cos d} + \frac{(\sin\theta_1 - \sin\theta_2)^2}{1 - \cos d} = 2\sin^2\theta_0, \tag{131}$$

$$Y = \frac{(\sin\theta_1 + \sin\theta_2)^2}{1 + \cos d} - \frac{(\sin\theta_1 - \sin\theta_2)^2}{1 - \cos d} = 2\sin^2\theta_0 \cos(d_1 + d_2)$$

From (131) $\sin^2\theta_0 = X/2$, $\sin^2\theta_0 \cos(d_1 + d_2) = Y/2$, and we can write (130) in the form:

$$\frac{S}{a} = d - (f/4)(Xd - Y \sin d)$$

$$+ (f^2/128) \begin{bmatrix} (16d^2 \cot d)X - (16d^2 \csc d)Y \\ + (2d + \sin 2d - 8d^2 \cot d)X^2 \\ + (8d^2 \csc d)XY - (2\sin 2d)Y^2 \end{bmatrix} \tag{132}$$

If we factor sin d out of every term of (132), we can write:

$$S = a \sin d\,[T - (f/4)(TX - Y) + (f^2/64)(A_0X + B_0Y + C_0X^2 + D_0XY + E_0Y^2)]$$

$T = d/\sin d$, $E_0 = -2\cos d$, $A_0 = -D_0E_0$, $C_0 = T - \tfrac{1}{2}(A_0 + E_0)$, (133)

$D_0 = 4T^2$, $B_0 = -2D_0$, d is the spherical distance between the points $P_1(\theta_1,\lambda_1)$ and $P_2(\theta_2,\lambda_2)$ given by some spherical formula as

$$\cos d = \sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2 \cos\Delta\lambda,\ \Delta\lambda = \lambda_2 - \lambda_1.$$

COMPARISON WITH AN EXISTING EXPANSION

On page 8, GIMRADA Research Note No. 11, E. M. Sodano, April 1963 [23] one finds the following formula:

$$\begin{aligned}\frac{S}{b_0} &= (1+f+f^2)\,\phi + a[(f+f^2)\sin\phi - (f^2/2)\,\phi^2 \csc\phi] \\ &+ m\left(-\frac{f+f^2}{2}\,\phi - \frac{f+f^2}{2}\sin\phi\cos\phi + \frac{f^2}{2}\,\phi^2\cot\phi\right) \\ &+ m^2\left(\frac{f^2}{16}\,\phi + \frac{f^2}{16}\sin\phi\cos\phi - \frac{f^2}{2}\,\phi^2\cot\phi - \frac{f^2}{8}\sin\phi\cos^3\phi\right) \\ &+ am\left(\frac{f^2}{2}\,\phi^2\csc\phi + \frac{f^2}{2}\sin\phi\cos^2\phi\right) - a^2\,(f^2/2)\sin\phi\cos\phi\end{aligned} \tag{134}$$

Now the correspondence between the parameters as used in (133) and those of Sodano are:

$$m(\text{Sodano}) = X/2,\ a(\text{Sodano}) = \tfrac{1}{4}(Y + X\cos d),\ \phi(\text{Sodano}) = d,\ b_0(\text{Sodano}) = a(1-f) \tag{135}$$

(a is equatorial radius, f the flattening).

If we substitute from (135) into (134), retaining terms to f^2 inclusive, we can write (134) as

$$\begin{aligned}\frac{S}{a} &= d - (f/4)\,(Xd - Y\sin d) \\ &+ (f^2/128)\left[\begin{aligned}&(16d^2\cot d)\,X - (16d^2\csc d)\,Y \\ &+ (2d + \sin 2d - 8d^2\cot d)\,X^2 \\ &+ (8d^2\csc d)\,XY - (2\sin 2d)\,Y^2\end{aligned}\right]\end{aligned} \tag{136}$$

Now comparing (132) and (136) find that the equations are identical which gives an independent check of Sodano's inverse formula.

COMPUTING FORM IN TERMS OF PARAMETRIC LATITUDE

Given on the reference ellipsoid the points $P_1(\theta_1, \lambda_1)$, $P_2(\theta_2, \lambda_2)$; P_2 west of P_1, west longitudes considered positive. (Geodetic latitudes are converted to parametric by $\tan\theta = (1-f) \cdot \tan\phi$ or an equivalent formula). Formulas (133) may be used as follows:

With $\theta_m = \tfrac{1}{2}(\theta_1+\theta_2)$, $\Delta\theta_m = \tfrac{1}{2}(\theta_2-\theta_1)$, $\Delta\lambda = \lambda_2 - \lambda_1$, $\Delta\lambda_m = \dfrac{\Delta\lambda}{2}$

let $\quad k = \sin\theta_m \cos\Delta\theta_m$, $K = \sin\Delta\theta_m \cos\theta_m$,

$H = \cos^2\Delta\theta_m - \sin^2\theta_m = \cos^2\theta_m - \sin^2\Delta\theta_m$,

$L = \sin^2\Delta\theta_m + H\sin^2\Delta\lambda_m = \sin^2 d/2$, $1 - L = \cos^2 d/2$,

$\cos d = 1 - 2L$, $h = \sin^2 d = 4L(1 - L)$, $U = 2k^2/(1 - L)$,

$V = 2K^2/L$, $X = U + V$, $Y = U - V$

$$T = (d/\sin d) = 1 + (1/6)h + (3/40)h^2 + (5/112)h^3 + (35/1152)h^4 + (63/2816)h^5 + \ldots$$

$$E_0 = -2\cos d,\ A_0 = -D_0E_0 = -4E_0T^2,\ D_0 = 4T^2,\ B_0 = -2D_0,\ C_0 = T - \tfrac{1}{2}(A_0 + E_0) \qquad (137)$$

$$S = a \sin d\ [T - (f/4)(TX - Y) + (f^2/64)(A_0X + B_0Y + C_0X^2 + D_0XY + E_0Y^2)]$$

$$\sin(\alpha_2 + \alpha_1) = (K \sin \Delta\lambda)/L,\ \sin(\alpha_2 - \alpha_1) = (k \sin \Delta\lambda)/(1 - L)$$

$$\tfrac{1}{2}(\delta\alpha_2 + \delta\alpha_1) = -(f/2)\ TH \sin(\alpha_1 + \alpha_2)$$

$$\tfrac{1}{2}(\delta\alpha_2 - \delta\alpha_1) = -(f/2)\ TH \sin(\alpha_2 - \alpha_1)$$

$$\alpha_{1-2} = \alpha_1 + \delta\alpha_1,\ \alpha_{2-1} = \alpha_2 + \delta\alpha_2.$$

The azimuth formulas of (137) are obtained by manipulation of expressions given on pages 126-128 of THE DISTANCE BETWEEN TWO WIDELY SEPARATED POINTS ON THE SURFACE OF THE EARTH, W. D. Lambert, Journal of the Washington Academy of Sciences, Vol. 32, No. 5, May 15, 1942, [13]. Note that in the distance and azimuth formulas of (137), the same quantities H, T, L, k, K are used.

Figure 22 in an example of the arrangements of equations (137) and computations for comparison with those of Figure 21, page 80. The results are:

True distance meters	Geodetic Latitude Fig. 21 δf	δf^2	Parametric Latitude Fig. 22 δf	δf^2
8,466,621.01	618.26	621.11	622.30	621.08
True Azimuths				
109° 57' 17".41		16".86		16".68
265° 37' 10".59		10".71		11".37

As was to be expected both approximations are adequate within any operational requirements. The coefficients A_0, B_0, C_0, D_0, E_0 of the parametric latitude form, Figure 22, are slightly less complicated than those of the geodetic form, Figure 21. But no conversion to parametric latitudes needs to be made for Figure 21. For purely geodetic computations further investigation needs to be made and it is suggested that computations be made using both forms against the computed lines contained in the revised issues of ACIC Reports 59 and 80, Sept. 1960 and December 1959.[12]

DISTANCE COMPUTING FORM, FORSYTH-ANDOYER-LAMBERT

TYPE APPROXIMATION WITH SECOND ORDER TERMS

$\tan\theta = 0.996609925 \tan\phi$

Clarke Spheroid 1866, a = 6,378,206.4 meters

$f/2 = 0.00169503765$, $f/4 = 0.000847518825$, $f^2/64 = 0.1795720390 \times 10^{-6}$

1 radian = 206,264.8062 seconds

ϕ_1 8° 58' 25".0 — 1. PANAMA — λ_1 74° 34' 24".0

ϕ_2 21 26 06.0 — 2. HAWAII — λ_2 158 01 33.0

$\theta_m = \frac{1}{2}(\theta_1+\theta_2)$ 15° 09' 22".644 — 2. Always west of 1. — $\Delta\lambda = \lambda_2 - \lambda_1$ 78° 27' 09".0

$\Delta\theta_m = \frac{1}{2}(\theta_2-\theta_1)$ 6 12 45.386 — θ_1 8° 56' 37".258 — $\Delta\lambda_m = \frac{1}{2}\Delta\lambda$ 39 13 34.5

θ_2 21 22 08.029

$\sin\theta_m$ + 0.26145290 — $\sin\Delta\theta_m$ + 0.10821810 — $\sin\Delta\lambda$ + 0.97975909

$\cos\theta_m$ + 0.96521623 — $\cos\Delta\theta_m$ + 0.99412718 — $\sin\Delta\lambda_m$ + 0.63238428

$k = \sin\theta_m \cos\Delta\theta_m$ + 0.25991743 — $K = \sin\Delta\theta_m \cos\theta_m$ + 0.10445387

$H = \cos^2\Delta\theta_m - \sin^2\theta_m = \cos^2\theta_m - \sin^2\Delta\theta_m$ + 0.91993122 — $1 - L$ + 0.62039926

$L = \sin^2\Delta\theta_m + H\sin^2\Delta\lambda_m$ + 0.37960074 — $\cos d = 1 - 2L$ + 0.24079852

d + 1.3276078324 — sin d + 0.97057512 — $T = d/\sin d$ + 1.367856856

$U = 2k^2/(1-L)$ + 0.2177857865 — $V = 2K^2/L$ + 0.0574846667 — $E = -2\cos d$ − 0.48159704

$X = U + V$ + 0.2752704532 — $Y = U - V$ + 0.1603011198 — $D = 4T^2$ + 7.484129512

$A = -DE = -4ET^2$ + 3.604334620 — $C = T - \frac{1}{2}(A+E)$ − 0.19351193 — $B = -2D$ − 14.968259024

$X(A + CX)$ + 0.977503686 — $Y(B + EY)$ − 2.411804017 — DXY + 0.330245911

$(TX - Y)$ + 0.216229457 — $\delta f = -(f/4)(TX - Y)$ -1.83259×10^{-4}

$T + \delta f$ + 1.367673597 — $S_1 = a\sin d\,(T + \delta f)$ 8,466,622.30 meters

$\Sigma = X(A + CX) + Y(B + EY) + DXY$ − 1.10405442 — $\delta f^2 = +(f^2/64)\,\Sigma$ -1.9826×10^{-7}

$T + \delta f + \delta f^2$ + 1.367673399 — $S_2 = a\sin d\,(T + \delta f + \delta f^2)$ 8,466,621.08 meters

$\sin(\alpha_2 + \alpha_1) = (K\sin\Delta\lambda)/L$ + 0.26959808 — $\alpha_1 + \alpha_2$ 375° 38' 25".266

$\sin(\alpha_2 - \alpha_1) = (k\sin\Delta\lambda)/(1-L)$ + 0.41047190 — $\alpha_2 - \alpha_1$ 155 45 55.864

$\frac{1}{2}(\delta\alpha_1 + \delta\alpha_2) = -(f/2)\,H\,T\sin(\alpha_2+\alpha_1)$ $-5.75032185 \times 10^{-4}$ — $\delta\alpha_1$ + $0.300473136 \times 10^{-3}$

$\frac{1}{2}(\delta\alpha_2 - \delta\alpha_1) = -(f/2)\,HT\sin(\alpha_2-\alpha_1)$ $-8.75505321 \times 10^{-4}$ — $\delta\alpha_2$ − $1.450537506 \times 10^{-3}$

	°	'	"			°	'	"
α_1	109	56	14.701		α_2	265	42	10.565
$\delta\alpha_1$		+ 1	01.977		$\delta\alpha_2$		− 4	59.195
α_{1-2}	109	57	16.678		α_{2-1}	265	37	11.370

$\alpha_{1-2} = \alpha_1 + \delta\alpha_1$ — $\alpha_{2-1} = \alpha_2 + \delta\alpha_2$

Figure 22

TRANSFORMATION FROM SECOND ORDER FORM IN GEODETIC LATITUDE TO SECOND ORDER IN PARAMETRIC

In terms of geodetic latitude, the equations corresponding to (132) are:

$$\frac{s}{a} = d' - (f/4)\,(X'd' - 3Y'\sin d')$$

$$+ (f^2/128)\,(AX' + BY' + CX'^2 + DX'Y' + EY'^2)$$

$$A = 64d' + 16d'^2 \cot d',\; B = -96 \sin d' - 16d'^2 \csc d', \tag{138}$$

$$C = -30d' - 15 \sin 2d' - 8d'^2 \cot d',$$

$$D = 48 \sin d' + 8d'^2 \csc d',\; E = 30 \sin 2d'$$

(See Equation (109), page 78.

Equation (132) is written here in the form:

$$\frac{s}{a} = d - (f/4)\,(Xd - Y \sin d) \tag{139}$$

$$+ (f^2/128)\,(A_0X + B_0Y + C_0X^2 + D_0XY + E_0Y^2)$$

$$A_0 = 16d^2 \cot d,\; B_0 = -16d^2 \csc d,\; C_0 = 2d + \sin 2d - 8d^2 \cot d,$$

$$D_0 = 8d^2 \csc d,\; E_0 = -2 \sin 2d$$

Now we should be able to find transformation equations of the form:

$$d' = d'(d, X, Y),\; X' = X'(X, Y, d),\; Y' = Y'(Y, X, d) \tag{140}$$

which when substituted in (138) should produce equations (139).

The definitions of X', Y' and X, Y are:

$$X' = 2 \sin^2\phi_0,\; X = 2 \sin^2\theta_0 \tag{141}$$

$$Y' = 2 \sin^2\phi_0 \cos(d_1' + d_2'),\quad Y = 2 \sin^2\theta_0 \cos(d_1 + d_2)$$

where ϕ_0, θ_0 are respectively geodetic, parametric latitude of the vertex of the great elliptic arc. From the equation $\tan\theta = (1 - f)\tan\phi$, or equivalent, we find:

$$\phi_0 = \theta_0 + f \sin\theta_0 \cos\theta_0\,(1 + f \cos^2\theta_0). \tag{142}$$

From the values indicated by Forsyth on page 120, of his treatise, to first order in f, extending the results to second order in f we find:

$$d' = d - (f/2)\,Y \sin d + (f^2/16)\,[4Y\,(X - 3) \sin d + (2Y^2 - X^2) \sin 2d] \tag{143}$$

and to first order in f,

$$\cos(d_1' + d_2') = \cos(d_1 + d_2) + f \cos d \sin^2\theta_0 - f \cos d \sin^2\theta_0 \cos^2(d_1 + d_2). \tag{144}$$

From (142), to first order in f, find

$$2 \sin^2\phi_0 = 2 \sin^2\theta_0\,(1 + 2f \cos^2\theta_0). \tag{145}$$

From (143), to first order in f, find

$$\sin d' = \sin d - (f/4)\, Y \sin 2d \tag{146}$$

From (141), (144), and (145) find

$$\begin{aligned} X' &= X + 2fX - fX^2 \\ Y' &= Y + 2fY - fXY + (f/2)\,(X^2 - Y^2) \cos d\,. \end{aligned} \tag{147}$$

Hence the transformations (140) are from (143), (146), and (147) the following:

$$\left\{\begin{aligned} d' &= d - (f/2)\, Y \sin d + (f^2/16)\;[4Y\,(X-3) \sin d + (2Y^2 - X^2) \sin 2d] \\ \sin d' &= \sin d - (f/4)\, Y \sin 2d \\ X' &= X + 2fX - fX^2 \\ Y' &= Y + 2fY - fXY + (f/2)\,(X^2 - Y^2) \cos d \end{aligned}\right. \tag{148}$$

Substution of the relations (148) into (138) produces equations (139), providing a second check of Sodano's formula for the inverse solution

The inverse of the transformations (148) which will carry (139) into (138) are:

$$\left\{\begin{aligned} d &= d' + (f/2)\, Y' \sin d' + (f^2/16)\,[4Y'(X'-1) \sin d' + (2Y'^2 - X'^2) \sin 2d'] \\ \sin d &= \sin d' + (f/4)\, Y' \sin 2d' \\ X &= X' - 2fX' + fX'^2 \\ Y &= Y' - 2fY' + fX'Y' + (f/2)\,(Y'^2 - X'^2) \cos d'. \end{aligned}\right. \tag{149}$$

DIFFERENCE FORMULAE FOR THE TWO SECOND ORDER FORMS

From equation (142) to second order in f, find

$$2 \sin^2\phi_0 = 2 \sin^2\theta_0\;(1 + 2f - 2f \sin^2\theta_0 + 3f^2 - 7f^2 \sin^2\theta_0 + 4f^2 \sin^4\theta_0), \tag{150}$$

and extending (144) to second order in f

$$\begin{aligned} \cos(d_1' + d_2') = {} & \cos(d_1 + d_2) + f \sin^2\theta_0 \cos d \sin^2(d_1 + d_2) \\ & - (f^2/2) \sin^2\theta_0 \sin^2(d_1 + d_2) \left[\begin{aligned} & \tfrac{1}{2} \sin^2\theta_0 \cos(d_1 + d_2) \\ & + \sin^2\theta_0 \cos d - (3/2) \cos d \\ & + (3/2) \sin^2\theta_0 \cos 2d \cos(d_1 + d_2) \end{aligned}\right] \end{aligned} \tag{151}$$

From equations (148), by factoring sin d out of every term of the expression for d', we can write:

$$d' = \sin d\,\{T - (f/2)\, Y + (f^2/8)\,[2Y(X-3) + (2Y^2 - X^2) \cos d]\} \tag{152}$$

Since we can write $X' = 2 \sin^2\phi_0$, $X = 2 \sin^2\theta_0$, $Y' = 2 \sin^2\phi_0 \cos(d_1' + d_2')$, $Y = 2 \sin^2\theta_0 \cos(d_1 + d_2)$ we have from (150) and then combining (150) and (151) (multiplying respective members together)

$$X' = X[1 + f(2 - X)\{1 + (f/2)(3 - 2X)\}] \tag{153}$$

$$Y' = Y[1 + f(2 - X)] + (f/2)(X^2 - Y^2)\cos d + (f^2/8)\left[\begin{array}{l} 4Y(2 - X)(3 - 2X) \\ \quad + (X^2 - Y^2)\{(11 - 5X)\cos d + Y(1 - 3\cos^2 d)\} \end{array}\right] \tag{154}$$

From Figure 22 we have

$$\begin{aligned} &X = 0.2752704532,\ Y = 0.1603011198, \\ &\sin d = 0.97057512,\ \cos d = 0.24079852, \\ &T = 1.367856856,\ f = 0.0033900753, \\ &f/2 = 0.00169503765,\ f^2/8 = 1.436576317 \times 10^{-6} \end{aligned} \tag{155}$$

Using the values from (155) to compute d', X', Y' from (152), (153), (154) find:

$$\begin{aligned} d' &= (0.97057512)(1.367856856 - 2.717164 \times 10^{-4} - 1.2634 \times 10^{-6}) \\ &= (0.97057512)(1.367583876) = 1.327342885; \\ X' &= (0.2752704532)(1.005871239) = 0.27688663; \\ Y' &= 0.160301120 + 9.37275 \times 10^{-4} + 2.0440 \times 10^{-5} + 4.068 \times 10^{-6} = 0.16126290. \end{aligned} \tag{156}$$

From Figure 21, $d' = 1.327342885$, $X' = 0.27688668$, $Y' = 0.16126298$ and comparing with the values from (156), we have a "flat" check for d', 5 in the eighth place for X' and 8 in the eighth place for Y'. Now the first significant figure of f^2 is 1 in the 5th decimal place and of f^3 is 4 in the 8th decimal place. If seven place tables are used, terms in f^2 are sufficient. If eight figure tables are used, as Peters trigonometric functions, there is some uncertainty in the 8th place of decimals.

Now the corresponding formulas for d, X, Y in the terms of d', X', Y' are found similarly to be, to second order terms in f inclusive;

$$\begin{aligned} d &= \sin d'\,\{T' + (f/2)Y' + (f^2/8)[2Y'(X' - 1) + (2Y'^2 - X'^2)\cos d']\} \\ X &= X'[1 + f(X' - 2)\{1 + (f/2)(2X' - 1)\}] \\ Y &= Y'[1 - f(2 - X')] - (f/2)(X'^2 - Y'^2)\cos d' \\ &\quad + (f^2/8)\left[\begin{array}{l} 4Y'(2 - X')(1 - 2X') \\ + (X'^2 - Y'^2)\{(5 - 3X')\,2\cos d' + Y'(1 - 3\cos^2 d')\} \end{array}\right] \end{aligned} \tag{157}$$

From Figure 21 we have

$$\begin{aligned} &X' = 0.276886675,\ Y' = 0.161262981, \\ &\sin d' = 0.97051129,\ \cos d' = 0.24105566 \\ &T' = 1.367673822. \end{aligned} \tag{158}$$

With the values of X', Y', $\sin d'$, $\cos d'$, T' from (158) and of f, f/2, $f^2/8$ from (155)

we find from (157) that

$$d = (0.97051129)\,(1.367673822 + 2.73347 \times 10^{-4} - 3.44 \times 10^{-7})$$

$$d = (0.97051129)\,(1.36794682) = 1.327607833$$

$$X = (0.276886675)\,(0.994162934) = 0.27527047 \qquad (159)$$

$$Y = 0.161262981 - 9.42015 \times 10^{-4} - 2.0700 \times 10^{-5} + 8.68 \times 10^{-7} = 0.16030113.$$

From(155). $X = 0.27527045$, $Y = 0.16030112$, and from Figure 22, $d = 1.327607832$. Comparing with (159) we have a difference in d of 1 in the 9th decimal place; in X and Y of 2 and 1 in the 8th decimal place respectively, which is within the computational uncertainties.

From (152), (153), (154), and (157) we can express the differences as functions of either set of variables:

$$\begin{aligned}
\Delta d = d' - d &= -(f/2)\,Y \sin d + (f^2/16)\,[4Y\,(X-3) \sin d + (2Y^2 - X^2) \sin 2d], \\
&= -(f/2)\,Y' \sin d' - (f^2/16)\,[4Y'(X'-1) \sin d' + (2Y'^2 - X'^2) \sin 2d'];
\end{aligned}$$

$$\begin{aligned}
\Delta X = X' - X &= fX(2-X)\,\{1 + (f/2)\,(3-2X)\}, \\
&= fX'(2-X')\,\{1 - (f/2)\,(1-2X')\}; \qquad (160)
\end{aligned}$$

$$\begin{aligned}
\Delta Y = Y' - Y &= fY\,(2-X) + (f/2)\,(X^2 - Y^2) \cos d \\
&\quad + (f^2/8) \begin{bmatrix} 4Y\,(2-X)\,(3-2X) \\ + (X^2 - Y^2)\,\{(11-5X) \cos d + Y\,(1 - 3\cos^2 d)\} \end{bmatrix}, \\
&= fY'(2-X') + (f/2)\,(X'^2 - Y'^2) \cos d' \\
&\quad - (f^2/8) \begin{bmatrix} 4Y'(2-X')\,(1-2X') \\ + (X'^2 - Y'^2)\,\{2(5-3X') \cos d' + Y'(1 - 3\cos^2 d')\} \end{bmatrix}.
\end{aligned}$$

SUMMARY OF DISTANCE COMPUTATIONS INVESTIGATION

As long as accuracy requirements remain within the range of the capabilities of the Andoyer-Lambert formulae, as exhibited in TABLE 3, they are quite adequate and it makes no difference if geographic latitudes are transformed to parametric latitudes first as far as accuracy requirements are concerned relative to hyperbolic electronic measuring systems. However, the formulae for geodetic azimuths are slightly more complicated in terms of geodetic latitude and some of the auxiliary quantities as chord length, dip of the chord, etc. are slightly less difficult to compute when expressed in terms of parametric latitude.

In order to arrange the computing in conformance with the Andoyer-Lambert formulae, equations (17), (48), (52), 56)), and (64) have been rearranged as follows to be expressible in common computational parameters:

The spherical length, d, is determined from formulae as given with Figure 16, $(d = d_1 + d_2)$;

$$\cot A = (\cos\theta_1 \tan\theta_2 - \sin\theta_1 \cos\Delta\lambda)/\sin\Delta\lambda$$

$$\cot B = (\cos\theta_2 \tan\theta_1 - \sin\theta_2 \cos\Delta\lambda)/\sin\Delta\lambda$$

$$\sin d = \cos\theta_1 \sin\Delta\lambda/\sin B = \cos\theta_2 \sin\Delta\lambda/\sin A;$$

these will compensate for any unfavorable triangle geometry.

The Andoyer-Lambert Formulae are taken in the form [13]

$$\delta d_r = -(f/8)\,(VQ^2/\sin^2 \tfrac{1}{2}d + UR^2/\cos^2 \tfrac{1}{2}d)$$

(1) $s = a(d_r + \delta d_r)$, $Q = \sin\theta_2 - \sin\theta_1$, $R = \sin\theta_1 + \sin\theta_2$.

$$V = d_r + \sin d, \quad U = d_r - \sin d,$$

With corresponding geodetic azimuths computed from

$$T = (f/2)\, d''/\sin d, \quad \delta A'' = T\cos^2\theta_2 \sin 2B,$$

(2) $\delta B'' = T\cos^2\theta_1 \sin 2A$; $g\alpha_{AB} = 180° - A + \delta A$; $g\alpha_{BA} = 180° + B - \delta B$

The Normal Section Azimuths may be written

(3) $\cot {}_n\alpha_{AB} = -(\cot A)/T_1 + (e^2 Q \cos\theta_1)/(\sin\Delta\lambda)T_1 \cos\theta_2$

$$\cot {}_n\alpha_{BA} = (\cot B/T_2 + (e^2 Q \cos\theta_2)/(\sin\Delta\lambda)T_2 \cos\theta_1$$

$$T_1 = (1 - e^2 \cos^2\theta_1)^{1/2} \quad T_2 = (1 - e^2 \cos^2\theta_2)^{1/2}$$

The chord length becomes

(4) $c = a\,(4 \sin^2 d/2 - e^2 Q^2)^{1/2}$

The angle of dip of the chord may be written

(5) $\beta = \arcsin\,[2b\,(\sin^2 d/2)/cT_1]$

b = semiminor axis of ellipsoid, c = chord length, $T_1 = (1 - e^2 \cos^2\theta_1)^{1/2}$.

The maximum separation of chord and arc becomes

(6) $H = (a^2/c)\,(1 - \cos \tfrac{1}{2}d)\,[4 \sin^2 d/2\,(\cos^2 d/2 - M) - e^2 Q^2]^{1/2}/\cos \tfrac{1}{2}d$

a = the semimajor axis of ellipsoid, c = chord length, $M = e^2 \sin\theta_1 \sin\theta_2$,

$Q = \sin\theta_2 - \sin\theta_1$, e = eccentricity of the spheroid.

The geographic coordinates of the point where maximum separation of chord and arc occurs

(7) $\tan\lambda = (\cos\theta_2 \sin\Delta\lambda)/(\cos\theta_1 + \cos\theta_2 \cos\Delta\lambda)$

$$\tan\phi = R/(0.996609925)\sqrt{4\cos^2 \tfrac{1}{2}d - R^2}$$

where $R = \sin\theta_1 + \sin\theta_2$.

Figure 23, shows the above formulae arranged in a computing form and the computations done over the line MOSCOW TO CAPE OF GOOD HOPE. See line No. 12, TABLE 1, and Figure 17.

COMPUTATIONS: GEODETIC DISTANCE AND AZIMUTHS, NORMAL SECTION AZIMUTHS, CHORD, ANGLE OF DIP, MAXIMUM SEPARATION, GEOGRAPHIC COORDINATES OF POINT OF MAXIMUM SEPARATION

Clarke 1866 Ellipsoid: $a = 6{,}378{,}206.4$ meters, $b = 6{,}356{,}583.8$ meters, $e^2 = 6.7686580 \times 10^{-3}$

$f/2 = 1.69503765 \times 10^{-3}$, $f/8 = 4.237594 \times 10^{-4}$, 1 radian = 206,264.8062 seconds

ϕ_1 +55° 45′ 19.″506	1 (A) MOSCOW	λ_1 −37° 34′ 15.″450
ϕ_2 −33° 56′ 03.″500	2 (B) CAPE OF GOOD HOPE	λ_2 −18° 28′ 41.″400
$\tan\phi_1$ +1.468 99522	2. Always West of 1.	$\Delta\lambda = \lambda_2 - \lambda_1$ +19° 05′ 34.″050
$\tan\phi_2$ −0.672 84157	$\tan\theta$ = 0.99660 9925 $\tan\phi$	$\sin\Delta\lambda$ +0.327 09901
$\tan\theta_1$ +1.464 01523	$\tan\theta_2$ −0.670 56059	$\cos\Delta\lambda$ +0.944 99007
$\sin\theta_1$ +0.825 75246	$\sin\theta_2$ −0.556 43719	$T_1 = (1-e^2\cos^2\theta_1)^{1/2}$ +0.998 92275
$\cos\theta_1$ +0.564 03269	$\cos\theta_2$ +0.830 55461	$T_2 = (1-e^2\cos^2\theta_2)^{1/2}$ +0.997 66269

$\cos^2\theta_1$ +0.318 13288 $\cos^2\theta_2$ +0.689 82096 A 164° 14′ 01.″416 B 10° 37′ 59.″721

$\cot A = (\cos\theta_1 \tan\theta_2 - \sin\theta_1 \cos\Delta\lambda)/\sin\Delta\lambda$ −3.541 88156 $\sin A$ +.271 71380 $\sin B$ +.184 52184

$\cot B = (\cos\theta_2 \tan\theta_1 - \sin\theta_2 \cos\Delta\lambda)/\sin\Delta\lambda$ +5.326 35275 $\sin 2A$ −.522 98282 $\sin 2B$ +.362 70661

$\sin d = \cos\theta_1 \sin\Delta\lambda/\sin B = \cos\theta_2 \sin\Delta\lambda/\sin A$ +0.99985202 d 90° 59′ 08.″549 d_r (radians) 1.58800018

$\cos d$ = −0.017 20300 $U = d_r - \sin d$ +.588 14816 $V = d_r + \sin d$ +2.58785220 $\sin \tfrac{1}{2}d$ +.71316302

$M = e^2 \sin\theta_1 \sin\theta_2$ −3.1128534×10⁻³ $Q = \sin\theta_2 - \sin\theta_1$ −1.38268965 $R = \sin\theta_1 + \sin\theta_2$ +0.26881527 $\cos\tfrac{1}{2}d$ +.700 99822

$\delta d_r = -(f/8)(VQ^2/\sin^2\tfrac{1}{2}d + UR^2/\cos^2\tfrac{1}{2}d)$ −.004 15887 (1) $S = a(d_r + \delta d_r)$ 10,102,066.780 meters

$T = (f/2)\, d''/\sin d$ 555.″289 $\delta A'' = T\cos^2\theta_2 \sin 2B$ +138.″935 $\delta B'' = T\cos^2\theta_1 \sin 2A$ −92.″388

$_n\alpha AB$ = arc cot $[-(\cot A)/T_1 + (e^2 Q \cos\theta_1)/(\sin\Delta\lambda)\, T_1 \cos\theta_2]$

$_n\alpha BA$ = arc cot $[(\cot B)/T_2 + (e^2 Q \cos\theta_2)/(\sin\Delta\lambda)\, T_2 \cos\theta_1]$

(2) Geodetic Azimuths: $_g\alpha AB = 180 - A + \delta A$ 15° 48′ 17.″519; $_g\alpha BA = 180 + B - \delta B$ 190° 39′ 32.″109

$_n\alpha AB$ 15° 49′ 57.″484; $_n\alpha BA$ 190° 41′ 29.″803 — Normal Section Azimuths (3)

(4) Chord: $c = a(4\sin^2\tfrac{1}{2}d - e^2Q^2)^{1/2}$ 9,068,422.241 m

Angle of dip of the chord (5) $\beta = \arcsin[2b(\sin^2 d/2)/cT_1]$ 45° 32′ 37.″538

Maximum separation of chord − arc: $H = (a^2/c)(1-\cos\tfrac{1}{2}d)[4\sin^2 d/2(\cos^2 d/2 - M) - e^2Q^2]^{1/2}/\cos\tfrac{1}{2}d$

(6) H_0 1,906,856.210 m

Geographic coordinates of point where maximum separation occurs: (7) $\tan\lambda = (\cos\theta_2 \sin\Delta\lambda)/(\cos\theta_1 + \cos\theta_2\cos\Delta\lambda)$ +.201 40402; λ +11° 23′ 14.″143

$\tan\phi = R/(0.996609925)\sqrt{4\cos^2\tfrac{1}{2}d - R^2}$ +.196 02673

λ_g −26° 11′ 01.″307 $(\lambda_1 + \lambda)$

ϕ 11° 18′ 16.″952

Andoyer-Lambert Approximation (Parametric latitude)

Figure 23.

Note in Figure 23 that two values of longitude are given, λ and λg. λ is the longitude associated with the point where maximum separation of chord and arc occurs but corresponding to the rectangular coordinate system as defined in say Figure 14. λg is the true geodetic longitude of the same point and is of course obtained by adding λ to λ_1 since λ_1 is negative.

While a continuous system based on either the great elliptic section as depicted by Figure 17, or the Forsyth-Andoyer-Lambert approximation, Figure 23, will provide all the information as indicated and accurate enough for hyperbolic electronic systems and any present operational requirements, the Forsyth-Andoyer-Lambert is probably to be preferred because of computational simplicity and existence of programs already operating for high speed computers. Should the need arise for accuracy of the order of 1 meter in distance and 1 second in azimuth over the ellipsoid, the extension to second order terms in the flattening provided by equations (110) or (137), will suffice.

Many formulae are available for geodetic lines and differential corrections are available for transforming elements such as geodetic azimuths to normal section azimuths, etc. [24]. Usually these are complicated, involve tabular material for a particular spheroid of reference, require extensive root computation, and accuracy depends on line length. For instance, Bomford says Rudoe's formulae for the reverse problem, are "Unnecessarily complex for general use," [21], page 108. Also they give "Normal Section" distances – not geodetic. The difference between the geodesic and the Normal Section distance is of 4th order in the eccentricity of the meridian ellipse [24].

Finally this investigation has raised the question as to whether either Andoyer or Lambert should share any credit for the first order approximation formula in terms of parametric latitude recognizable intact in Forsyth's 1895 paper. While Forsyth had an erroneous second order term to the same expansion in terms of geodetic latitude, his first order term was correct and he thus had both so-called Andoyer-Lambert formulae. Gougenheim apparently had in 1950 the first correct expansion in print in terms of geodetic latitude which included the second order terms in the flattening.

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APPENDIX 1

Example of
Computations of Loran Lines
of Position (Plane Approximation)

Intersections of Loran Lines of Position

(Plane Approximation)

P. D. Thomas, Mathematician

Consider the hyperbolic system as shown in Figure 24. The hyperbolic locus with foci F, F′ has for equation

$$(c^2 - a^2)\, x^2 - a^2y^2 = a^2\,(c^2 - a^2),\ (e = \frac{c}{a} > 1) \tag{1}$$

As a varies ($a < c$) all the hyperbolas with the fixed foci F, F′ (which are 2c apart) are generated.

The hyperbolic locus with the fixed foci F, F″ when referred to the same coordinate system as (1), has for equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,\ (e = d/b > 1). \tag{2}$$

where one may first compute $\tau = b^2 - d^2$, $\mu = d\cos\alpha$, $\nu = d\sin\alpha$, $S = \tau - c\mu$, and then $A = \mu^2 - b^2$, $B = 2\mu\nu$, $C = \nu^2 - b^2$, $D = 2(\tau\mu - cA)$, $E = 2S\nu$, $F = S^2 - b^2c^2$.

As b varies ($b < d$) all the hyperbolas with the fixed foci F, F″ (which are 2d apart) are generated.

The respective pairs of constants c, a; d, b for each hyperbola are related to the fundamental constants of a Loran line by

$$c = kB_1/2,\ a = kV_1/2;\ d = kB_2/2,\ b = kV_2/2 \tag{2.1}$$

where $v_i = t_i$, t_i is the time difference, v_i is the difference of light microseconds, B_i is the length (measured in light microseconds) of the direct line (baseline) between two Loran stations. k is the length of a light microsecond in the linear units in which x and y are expressed.[1]

Since five distinct points determine a conic uniquely, two conics can have at most four points in common. For the hyperbolas (1) and (2) there will always be four real points of intersection except when F′, F, F″ are collinear ($\alpha = 0$) and then there will be two.

ALGEBRAIC SOLUTIONS

I. If equations (1) and (2) are solved simultaneously for x one obtains the quartic equation

$$x^4 + Hx^3 + Jx^2 + Mx + N = 0 \tag{3}$$

where one may first compute $G = c^2 - a^2$, $\beta_0 = CG + Aa^2$, $\omega = F - CG$, $\delta = BEG$, $\gamma = a^2B^2 - E^2$, $L = \beta_0{}^2 - G\,B^2a^2$, and then $H = 2a^2\,(D\beta_0 - \delta)/L$, $J = a^2\,(a^2D^2 + 2\beta_0\,\omega + G\gamma)/L$,

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[1] Loran; Pierce, McKenzie, Woodward; McGraw Hill, 1948, pages 52, 53, 174.

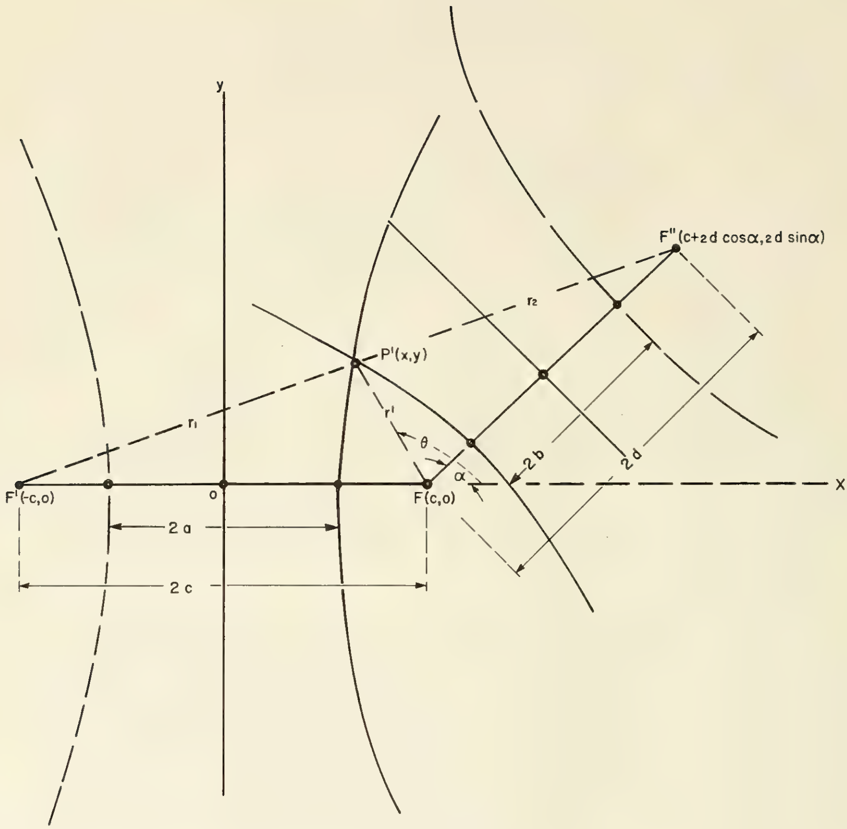

Figure 24. Two plane hyperbolas with a common focus.

$M = 2a^4(D\omega + \delta)/L$, $N = a^4(\omega^2 + GE^2)/L$. The corresponding values of y are then given by $y = \pm [G(x^2 - a^2)]^{1/2}/a$.

Equation (3) may be solved by the standard algebraic method[2] or by any of a number of numerical techniques.[3]

II. Again, if equations (1) and (2) are written in the forms $x^2 - Qy^2 = a^2$, $x^2 + Uxy + Vy^2 + Wx + Ry + T = 0$, where $Q = a^2/(c^2 - a^2)$, $U = B/A$, $V = C/A$, $W = D/A$, $R = E/A$, $T = F/A$ and these forms of the equations solved simultaneously with the line of slope m through the common focus $F(c,o)$ whose equation is $y = m(x - c)$, one obtains the two equations:

$$(Qm^2 - 1)\, x^2 - 2cQm^2x + (a^2 + c^2Qm^2) = 0, \tag{4}$$

$$(1 + Um + Vm^2)\, x^2 + [W + (R - cU)m - 2cVm^2]\, x + (c^2Vm^2 - cRm + T) = 0.$$

The resultant of the quadratic equations (4) is the condition that they have the same solutions or correspondingly that the parameter m will be restricted to those values for the points common to the hyperbolas (1) and (2).[4]

The resultant of the quadratics $a_0x^2 + a_1x + a_2 = 0$, $b_0x^2 + b_1x + b_2 = 0$ is given by

$$(a_0b_2 - b_0a_2)^2 + (b_1a_2 - a_1b_2)\,(a_0b_1 - a_1b_0) = 0. \tag{5}$$

From (4) $a_0 = Qm^2 - 1$, $a_1 = -2cQm^2$, $a_2 = a^2 + c^2Qm^2$, $b_0 = 1 + Um + Vm^2$, $b_1 = [W + (R - cU)\,m - 2cVm^2]$, $b_2 = c^2Vm^2 - cRm + T$, and these values placed in (5) lead to the quartic equation

$$k_1m^4 + k_2m^3 + k_3m^2 + k_4m + k_5 = 0, \tag{6}$$

where with $G = c^2 - a^2$, $\Omega = (a^2 + c^2)\, V + O\ (c^2 - T)$, $\theta_0 = R + cU$, $\phi = c^2 + cW + T$, $\eta = R - cU$, $\xi = a^2U - cR$, $\rho = a^2 - T$, $\rho' = a^2 + T$ one finds: $k_1 = (GV + \phi Q)^2 - a^2\theta_0^2$, $k_2 = 2[\xi\Omega + 2\eta ca^2V + a^2RQ \cdot (W + 2c) + c^2QU(cW + 2T)]$, $k_3 = \xi^2 - a^2\eta^2 + 2\rho'\Omega + W[4a^2cV + 2c\rho Q - a^2W]$, $k_4 = 2(\rho'\xi - a^2W\eta)$, $k_5 = \rho'^2 - a^2W^2$.

Again the solutions of (6) may be found by well known algebraic or numerical methods. The values of m obtained are of course the slopes of the lines through $F(c,o)$ and the points of intersection of the hyperbolas (1) and (2).

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[4] College Algebra, H. B. Fine, Page 512.

POLAR SOLUTION

The following procedure involves tables of the trigonometric functions but no root extraction. First express the equations of (1) and (2) in polar form both referred to the common focus F(c,o), and the corresponding rectangular coordinates in terms of the polar parameters. Find for equation (1)

$$r_a = \frac{c^2 - a^2}{\pm a - c \cos \theta} \quad (c>a) \quad \text{(see equation (3) PLANE, page 37 with } R = r_a,\ \beta = \theta)$$

$$x = c + r_a \cos \theta,\ y = r_a \sin \theta \tag{7}$$

and for equation (2)

$$r_b = \frac{(d^2 - b^2)\,[d \cos (\theta - \alpha) \pm b]}{d^2 \cos^2 (\theta - \alpha) - b^2} \quad (d > b)$$

$$x = c + r_b \cos \theta,\ y = r_b \sin \theta \tag{8}$$

Since (7) and (8) express the two hyperbolas in polar form with respect to the same pole F(c,o), a common focus of the two loci, it is clear (see Figure 24) that at a point of intersection $P'(x,y)$ the two values r_a and r_b are equal to a common value r' for a common value of θ and the distances to P' from F' and F'' are then given by $r_1 = r' + 2a$, $r_2 = r' + 2b$.

Equating the values of r_a, r_b from (7) and (8) one obtains

$$r' = \frac{c^2 - a^2}{\pm a - c \cos \theta} = \frac{d^2 - b^2}{d \cos (\theta - \alpha) \pm b} \tag{9}$$

and since c, d, α are constants, (9) is a relation between the parameters a, b, and θ. That is given any two of the three the third may be found from (9).

Consider a and b given. First write (9) in the form

$$\frac{d \cos (\theta - \alpha) \mp b}{\pm a - c \cos \theta} = \frac{d^2 - b^2}{c^2 - a^2} = K, \text{ whence}$$

$$(d \cos \alpha + cK) \cos \theta + (d \sin \alpha) \sin \theta = \pm aK \pm b. \tag{10}$$

The solution of the trigonometric equation (10) is

$$\theta_i = \beta + \gamma_i$$

$$\tan \beta = (d \sin \alpha)/(d \cos \alpha + cK) \qquad (i = 1,2,3,4)$$

$$\cos \gamma_i = (\pm aK \pm b) \sin \beta / d \sin \alpha. \tag{11}$$

From (11) it is seen that in general there will be four angles (γ_i), and thus four values

of θ_i, four values of r_i' from (9) and four sets of rectangular coordinates from $x_i = c + r_i' \cos\theta_i$, $y_i = r_i' \sin\theta_i$ $(i = 1,2,3,4)$ (12)

and for each point of intersection two of the additional distances

$$r_i = r_i' \pm 2b,\ r_{i+4} = r_i' \pm 2a \qquad (i = 1,2,3,4). \tag{13}$$

A procedure for using equations (9) through (13) will be described and used for two examples. Since a,b,c,d,α will be given, first compute $K = (d^2 - b^2)/(c^2 - a^2)$, $\mu = d\cos\alpha$, $\nu = d\sin\alpha$, $\tan\beta = \nu/(\mu + cK)$.

From $\tan\beta$, using tables, find β and $\sin\beta$. Then compute

$\cos\gamma_i = (\pm aK \pm b)\sin\beta/\nu$ $(i = 1,2,3,4)$, and

$\theta_i = \beta + \gamma_i$ $(i = 1,2,3,4)$. Next compute

$$r_i' = \frac{c^2 - a^2}{\pm a - c\cos\theta_i} = \frac{d^2 - b^2}{d\cos(\theta_i - \alpha) \pm b} \quad i = 1,2,3,4$$

choosing the proper value (with respect to sign) of $\pm a$, $\pm b$ in each member which will make them equal and positive for each value of θ_i. Now the rectangular coordinates may be computed from $x_i = c + r_i'\cos\theta_i$, $y_i = r_i'\sin\theta_i$. Useful checks are provided at this point by the relations $(x_i - c)^2 + y_i^2 = r_i'^2$ and by $\Sigma x_i = -H$ from equation (3). $H = 2a^2(D\beta_0 - \delta)/L$, $\beta_0 = CG + Aa^2$, $\delta = BEG$, $L = \beta_0^2 - GB^2a^2$, $G = c^2 - a^2$, $A = \mu^2 - b^2$, $B = 2\mu\nu$, $C = \nu^2 - b^2$, $D = 2(\tau\mu - cA)$, $E = 2S\nu$, $\tau = b^2 - d^2$, $S = \tau - c\mu$. Finally compute the additional distances $r_i = r_i' \pm 2b$, $r_{i+4} = r_i' \pm 2a$. $(i = 1,2,3,4)$.

<u>Example 1.</u> Let $c = d = 2$, $a = b = 1$, $\alpha = 45°$. $\sin\alpha = \cos\alpha = \sqrt{2}/2$.

$K = (d^2 - b^2)/(c^2 - a^2) = 1$. $\nu = \mu = 2\,(0.70710678) = 1.41421356$.

$\tan\beta = \nu/(\mu + cK) = (1.41421356)/(3.41421356) = 0.41421356$.

$\beta = 22°30'$, $\sin\beta = 0.38268343$.

$\cos\gamma_i = (\pm aK \pm b)(\sin\beta/\nu) = (\pm 1 \pm 1)(0.27059805) = \pm(0.54119610),\ 0.$

$0 < \gamma_i < 2\pi$.

$\gamma_i = 57°\,14'\,05''.666,\ 90°,\ 122°\,45'\,54''.334,\ 270°$

$\theta_i = \beta + \gamma_i$, $\theta_1 = 79°\,44'\,05'.666$, $\theta_2 = 112°\,30'$, $\theta_3 = 145°\,15'\,54''.334$, $\theta_4 = 292°\,30'$

$$r_i' = \frac{3}{\pm 1 - 2\cos\theta_i} = \frac{3}{2\cos(\theta_i - 45) \pm 1}$$. (Choose the proper value of ± 1 in each member which will make them equal and positive for each value of θ_i. If this cannot be done the values of θ_i may be in error.) The work may be arranged in table form as follows:

Table 1.

θ_i	$\theta_i - 45$	$\sin\theta_i$	$\cos\theta_i$	$\cos(\theta_i - 45)$	r_i'
$79°\ 44'\ 05''.666$	$34°\ 44'\ 05''.666$	0.98399379	0.17820275	0.82179706	4.6613215
112 30	67 30	0.92387953	−0.38268343	0.38268343	1.6993635
145 15 54.334	100 15 54.334	0.56978031	−0.82179706	−0.17820275	4.6613215
292 30	247 30	−0.92387953	0.38268343	−0.38268343	12.785918

$x_i = 2 + r_i'\cos\theta_i$	$y_i = r_i'\sin\theta_i$	$r_i = r_i' \pm 2$	$r_{i+4} = r_i' \pm 2$
2.8306603	4.5867114	$r_1 = 2.6613215$	$r_5 = 6.6613215$
1.3496817	1.5700072	$r_2 = 3.6993635$	$r_6 = 3.6993635$
−1.8306603	2.6559292	$r_3 = 6.6613215$	$r_7 = 2.6613215$
6.8929590	−11.812648	$r_4 = 14.785918$	$r_8 = 14.785918$

Checks were computed but are not shown here. Figure 25 shows the results ot Table 1 graphically.

Example 2. Let $c = 3$, $a = d = 2$, $b = 1$, $\alpha = 30°$. $\sin\alpha = ½$, $\cos\alpha \doteq \sqrt{3}/2$

$K = 0.6$, $\tan\beta = 1/(\sqrt{3} + 1.8) = 1/(3.5320508) = 0.28312164$, $\nu = 1$, $\mu = \sqrt{3}$.

$\beta = 15°\ 48'\ 28''.676$. $\sin\beta = 0.27241402$, $\cos\gamma_i = \dfrac{(\pm 1.2 \pm 1)}{2}\ (0.54482804)$

$\cos\gamma_i = \pm\ (1.1)\ (0.54482804),\ \pm\ (0.1)\ (0.54482804)$

$\cos\gamma_i = \pm\ 0.59931084,\ \pm\ 0.054482804$

$\gamma_i = 53°\ 10'\ 46''.000,\ 86°\ 52'\ 36''.550,\ 126°\ 49'\ 14''.000,\ 273°\ 07'\ 23''.450$

$\theta_i = \beta + \gamma_i$, $\theta_1 = 68°\ 59'\ 14''.676$, $\theta_2 = 102°\ 41'\ 05''.226$, $\theta_3 = 142°\ 37'\ 42''.676$

$\theta_4 = 288°\ 55'\ 52''.126$. $r_i' = \dfrac{5}{\pm 2 - 3\cos\theta_i} = \dfrac{3}{2\cos(\theta_i - 30) \pm 1}$. The work is arranged in the following table:

Table 2

θ_i	$\theta_i - 30$	$\sin\theta_i$	$\cos\theta_i$	$\cos(\theta_i - 30)$	r_i'
° ' "	° ' "				
68 59 14.676	38 59 14.676	0.93350166	0.35857308	0.77728423	5.40961166
102 41 05.226	72 41 05.226	0.97559289	−0.21958714	0.29762840	1.88057496
142 37 42.676	112 37 42.676	0.60698032	−0.79471687	−0.38475484	13.015729
288 55 52.126	258 55 52.126	−0.94590914	0.32443167	−0.19198850	4.86994806

$x_i = 3 + r_i' \cos\theta_i$	$y_i = r_i' \sin\theta_i$	$r_i = r_i' \pm 2$	$r_{i+4} = r_i' \pm 4$	$\tan\theta_i$
4.93974111	5.04988146	r_1 = 3.40961166	r_5 = 9.40961161	2.60337906
2.58704992	1.83467556	r_2 = 3.88057496	r_6 = 5.88057496	− 4.4428508
− 7.34381941	7.90029135	r_3 = 15.015729	r_7 = 9.015729	− 0.76376927
4.57996538	− 4.60652838	r_4 = 6.86994806	r_8 = 8.86994806	− 2.91558822

Checks of the computations of Table 2 were made as follows:

1. Using $(x_i - 3)^2 + y_i^2 = r_i^2$ and values from Table 2:

$(x_i - 3)^2$	y_i^2	$(x_i - 3)^2 + y_i^2$	r_i^2
3.762 59557	25.501 30276	29.263 89833	29.263 89831
0.170 52777	3.366 03441	3.536 56218	3.536 56218
106.994 59999	26.414 60341	169.409 20340	169.409 20140
2.496 29060	21.220 10372	23.716 39432	23.716 39410

2. From the formulas of (2) and (3) find $A = 2$, $B = 2\sqrt{3}$, $C = 0$, $D = -6(\sqrt{3}+2)$, $E = -6(\sqrt{3}+1)$, $F = 9(2\sqrt{3}+3)$, $\delta = BEG = -60(\sqrt{3}+3)$, $\beta_0 = a^2A + CG = 8$, $L = \beta_0^2 - a^2 GB^2 = -11 \times 2^4$
$H = -2^3[-48(\sqrt{3}+2) + 60(\sqrt{3}+3)]/11 \times 2^4, = \mp (2/11)[26.1961524] = -4.76293680$.
From Table 2, $\Sigma x_i = 4.76293700 = -H = 4.76293680$. Again computing N from equations (3), find $N = -429.826515$. From Table 2 find $\Pi x_i \equiv -429.826494$ and $\Pi x_i = N$.

3. From equation (6), compute the quantities:
$U = B/A = \sqrt{3}$, $V = C/A = 0$, $W = D/A = -3(\sqrt{3}+2)$, $R = E/A = -3(\sqrt{3}+1)$, $T = F/A = 9(2\sqrt{3}+3)/2$, $\phi = c^2 + cW + T = 9/2$, $\theta_0 = R + cU = -3$, $\rho' = a^2 + T = \frac{1}{2}(18\sqrt{3}+35)$,
$Q = a^2/(c^2 - a^2) = 4/5$, $k_1 = (GV + \phi Q)^2 - a^2\theta_0^2 = -2^6 3^2/5^2$, $k_5 = \rho'^2 - a^2W^2 = +(1189 + 684\sqrt{3})/2^2$.
Now from equation (6), $\Pi m_i = \Pi \tan\theta_i = k_5/k_1 = -5^2(1189 + 684\sqrt{3})/2^8 3^2 = -25.756540$.
Now forming $\Pi \tan\theta_i$ from the values in Table 2, find

$$\Pi \tan\theta_i = -25.756539.$$

Figure (26) depicts the solution graphically.

SUMMARY REMARKS (Plane Approximation)

While the formulas (9) through (13) are convenient for hand computing, since no root extraction is involved, the use of trigonometric tables may make it unsuitable for larger machine coding and computation, and it may be better to use the algebraic solution, equation (3). If the algebraic solution is to be used, the number of significant figures to be retained in the coefficients of the resulting quartic, equation (3), will have to be considered relative to the number of significant figures required in the rectangular coordinates of the intersections points.

If solutions only above the base line, $F' F''$, are desired (see Figure 24), then in the trigonometric solution, equations (9) – (13), θ should be limited to $\pi > \theta > \alpha$.

Note that the parameters a and b of the two families of confocal hyperbolas are related to the fundamental constants of a Loran line by the relations (2.1).

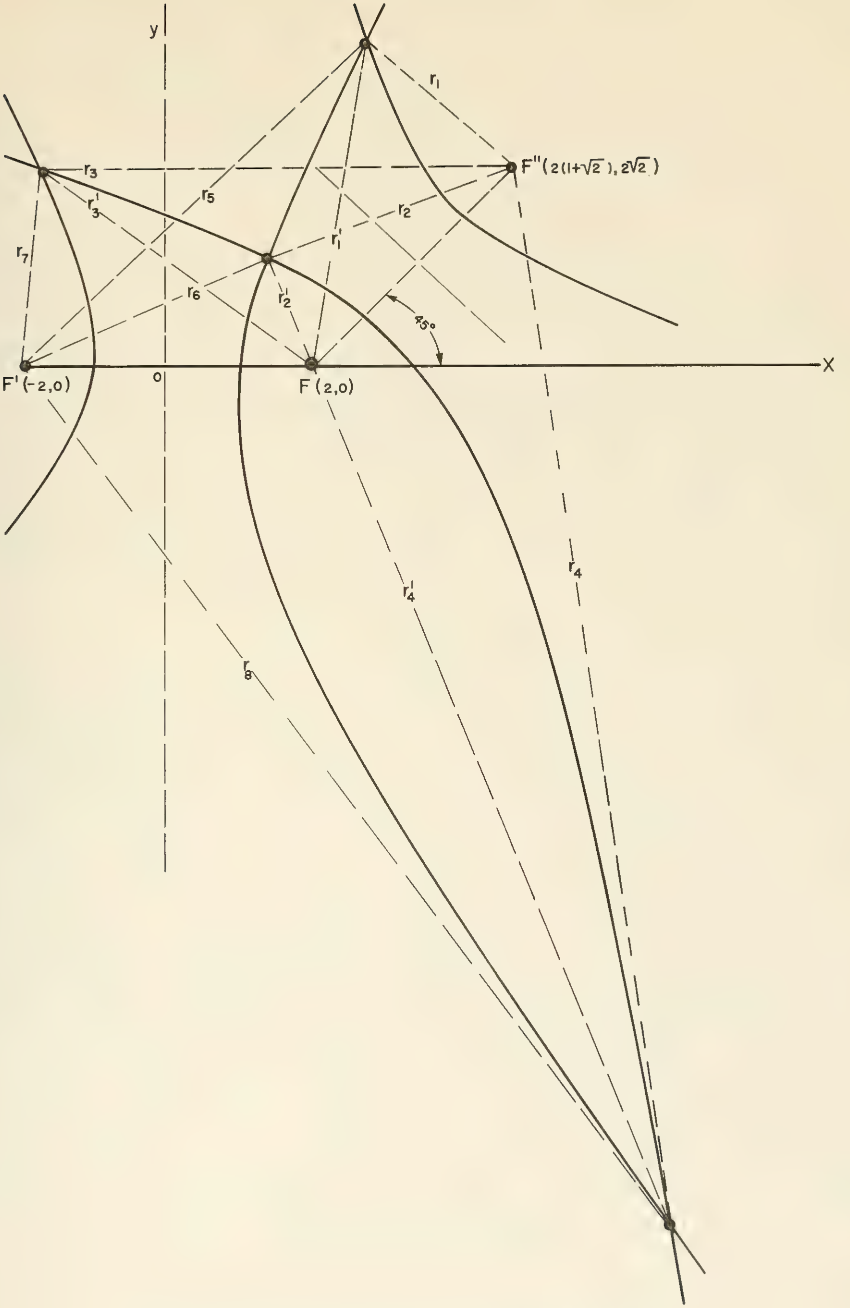

Figure 25. Intersection of plane hyperbolas. Example 1.

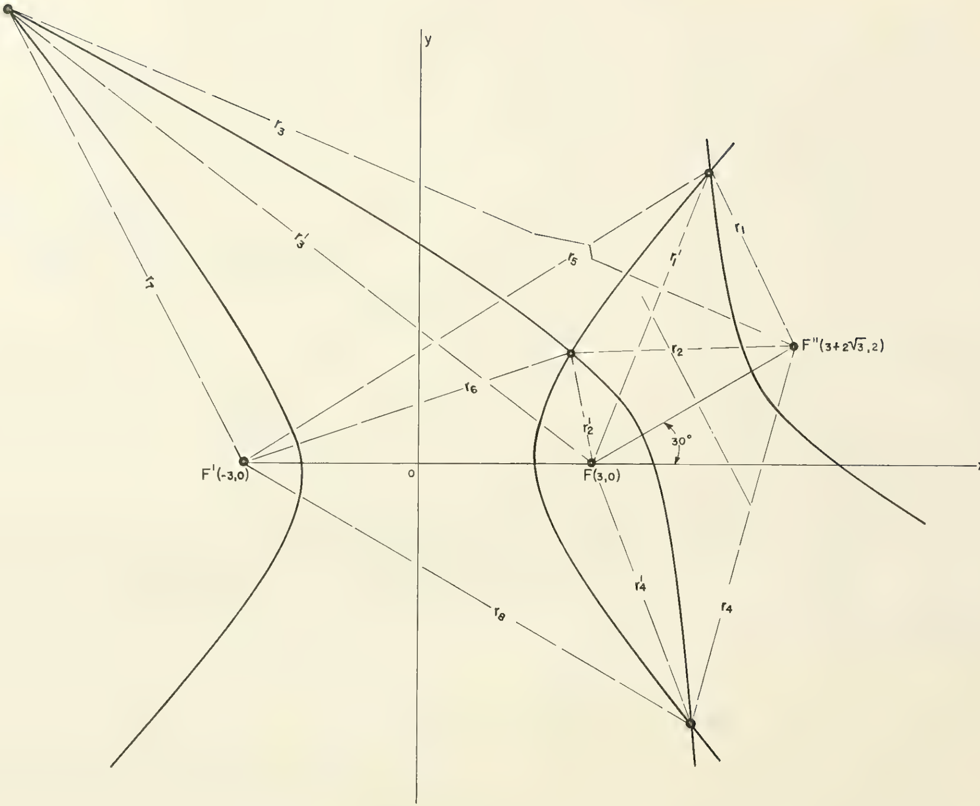

Figure 26. Intersection of plane hyperbolas. Example 2.

APPENDIX 2

Computations
Using Andoyer-Lambert
First Order Formulae Without Conversion
to Parametric Latitude

DISTANCE COMPUTING FORM, ANDOYER-LAMBERT APPROXIMATION

(No conversion to parametric latitudes)

Clarke Spheroid 1866 a = 6,378,206.4 meters

f/2 = 0.00169503765, f/4 = 0.000847518825

1 radian = 206,264.8062 seconds

ϕ_1 40° 30′ 37″.757	1. Origin	λ_1 17° 19′ 43″.280
ϕ_2 40 00 00.000	2. Terminus	λ_2 18 00 00.000
$\sin\phi_1$ .64958723	2. West of 1.	$\Delta\lambda=\lambda_2-\lambda_1=$ 40 16.720
$\cos\phi_1$ .76028707	$\sin\phi_2$ .64278761	$\sin\Delta\lambda$ .01171632
$\tan\phi_1$ .85439731	$\cos\phi_2$ .76604444	$\cos\Delta\lambda$ .99993136

$\tan\phi_2$ .83909963 $\cos d=\sin\phi_1\sin\phi_2+\cos\phi_1\cos\phi_2\cos\Delta\lambda$ .99992033

$M=\cos\phi_1\tan\phi_2-\sin\phi_1\cos\Delta\lambda$ −.01158604 $\cot A=\frac{M}{\sin\Delta\lambda}$ −.98888047

$N=\cos\phi_2\tan\phi_1-\sin\phi_2\cos\Delta\lambda$ +.01176282 $\cot B=\frac{N}{\sin\Delta\lambda}$ +1.00396882

$\sin d=\frac{\cos\phi_1\sin\Delta\lambda}{\sin B}$ +.01262251 $\sin A$ .71104900 A 134° 40′ 46″.816

$=\frac{\cos\phi_2\sin\Delta\lambda}{\sin A}$ +.01262251 $\sin B$ .70570498 B 44 53 11.497

d 43′ 23″.667

$K=(\sin\phi_1-\sin\phi_2)^2$ 4.62×10^{-5} $H=(d+3\sin d)/(1-\cos d)$ 633.744947

$L=(\sin\phi_1+\sin\phi_2)^2$ 1.67023273 $G=(d-3\sin d)/(1+\cos d)$ −.0126228028

$\delta d=-(f/4)(HK+GL)$ -6.9463×10^{-6} $s=a(d+\delta d)$ 80,467.388 meters

d (radians) .0126229382 s 43.4489 n.m.

$d+\delta d$ (rad) .01261599 $T=d/\sin d$ 1.000033576

2A 269° 21′ 33″.632 2B 89° 46′ 22″.994

sin 2A −.99993749 sin 2B .99999216

$U=(f/2)\cos^2\phi_1\sin 2A$ -9.79732265×10^{-4} $V=(f/2)\cos^2\phi_2\sin 2B$ $+9.94681111\times10^{-4}$

VT $+9.947145\times10^{-4}$ UT -9.7976516×10^{-4}

$\delta A=VT-U$ +.0019744468 $\delta B=-UT+V$ +.0019744627

$+\delta A$	+	6	47.259	$+\delta B$	+	6	47.262
$-A$	−134	40	46.816	$+B$	+44	53	11.497
+ 180				+ 180			
α_{1-2}	45°	26′	00″.443	α_{2-1}	224°	59′	58″.759

$\alpha_{1-2}=\alpha_{AB}=180°-A+\delta A$ $\alpha_{2-1}=\alpha_{BA}=180°+B+\delta B$

Line No. 1 (See Tables 1,2 - pages 65,66)

DISTANCE COMPUTING FORM, ANDOYER-LAMBERT APPROXIMATION

(No conversion to parametric latitudes)

Clarke Spheroid 1866 a = 6,378,206.4 meters

f/2 = 0.00169503765, f/4 = 0.000847518825

1 radian = 206,264.8062 seconds

ϕ_1 9° 59′ 48.349″ 1. Origin λ_1 16° 31′ 55.877″

ϕ_2 10 00 00.000 2. Terminus λ_2 18 00 00.000

sin ϕ_1 .17359255 2. West of 1. $\Delta\lambda = \lambda_2 - \lambda_1 =$ 1 28 04.123

cos ϕ_1 .98481756 sin ϕ_2 .17364818 sin $\Delta\lambda$ .02561535

tan ϕ_1 .17626874 cos ϕ_2 .98480775 cos $\Delta\lambda$ .99967188

tan ϕ_2 .17632698 cos d = sin ϕ_1 sin ϕ_2 + cos ϕ_1 cos ϕ_2 cos $\Delta\lambda$ .99968177

M = cos ϕ_1 tan ϕ_2 − sin ϕ_1 cos $\Delta\lambda$ +.00011432 cot A = $\frac{M}{\sin \Delta\lambda}$ +.00446295

N = cos ϕ_2 tan ϕ_1 − sin ϕ_2 cos $\Delta\lambda$ −.00000038 cot B = $\frac{N}{\sin \Delta\lambda}$ −.00001483

sin d = $\frac{\cos \phi_1 \sin \Delta\lambda}{\sin B}$ .02522645 sin A .99999004 A 89° 44′ 39.457″

= $\frac{\cos \phi_2 \sin \Delta\lambda}{\sin A}$ .02522645 sin B 1.00000000 B 90 00 03.060

K = (sin ϕ_1 − sin ϕ_2)^2 3.1 X 10^{-9} H = (d + 3 sin d)/(1 − cos d) 317.092888

L = (sin ϕ_1 + sin ϕ_2)^2 .12057612 G = (d − 3 sin d)/(1 + cos d) −.025229129

δd = −(f/4) (HK + GL) +3.0410 X 10^{-6} s = a (d + δd) 160,935.945 meters

d (radians) .025229122 s 86.8984 n.m.

d + δd (rad) .025232163 T = d/sin d 1.0001059 28

2A 179° 29′ 18.″914 2B 180° 00′ 06.″120

sin 2A +.00892572 sin 2B −.00002967

U = (f/2) cos$^2\phi_1$ sin 2A +1.467352 X 10^{-5} V = (f/2) cos $^2\phi_2$ sin 2B −4.878 X 10^{-8}

VT −4.878 X 10^{-8} UT +1.46751 X 10^{-5}

δA = VT − U −1.4722 X 10^{-5} δB = −UT + V −1.4724 X 10^{-5}

+ δA − 03.037 + δB − 03.037

− A − 89 44 39.457 + B + 90 00 03.060

+ 180 + 180

α_{1-2} 90° 15′ 17.″506 α_{2-1} 270° 00′ 00.″023

$\alpha_{1-2} = \alpha_{AB}$ = 180° − A + δA $\alpha_{2-1} = \alpha_{BA}$ = 180° + B + δB

Line No. 2 (See Tables 1,2 - pages 65,66)

DISTANCE COMPUTING FORM, ANDOYER-LAMBERT APPROXIMATION

(No conversion to parametric latitudes)

Clarke Spheroid 1866 a = 6,378,206.4 meters

f/2 = 0.00169503765, f/4 = 0.000847518825

1 radian = 206,264.8062 seconds

ϕ_1 69° 48′ 05.″701 1. Origin λ_1 9° 37′ 28.″637

ϕ_2 70° 00′ 00.″000 2. Terminus λ_2 18° 00′ 00.″000

$\sin\phi_1$ .938 .50257 2. West of 1. $\Delta\lambda = \lambda_2 - \lambda_1 =$ 8 22 31.363

$\cos\phi_1$ .345 2722.6 $\sin\phi_2$ .93969262 $\sin\Delta\lambda$ .145 65790

$\tan\phi_1$ 2.718 15224 $\cos\phi_2$ .342 02014 $\cos\Delta\lambda$ .989 33502

$\tan\phi_2$ 2.747 47742 $\cos d = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos\Delta\lambda$ .998 73458

$M = \cos\phi_1 \tan\phi_2 - \sin\phi_1 \cos\Delta\lambda$ +.020 13428 $\cot A = \dfrac{M}{\sin\Delta\lambda}$ +.138 22992

$N = \cos\phi_2 \tan\phi_1 - \sin\phi_2 \cos\Delta\lambda$ −.000 00801 $\cot B = \dfrac{N}{\sin\Delta\lambda}$ −.0000 5499

$\sin d = \dfrac{\cos\phi_1 \sin\Delta\lambda}{\sin B}$ +.05029163 $\sin A$ +0.990 58101 A 82° 07′ 47.″577

$= \dfrac{\cos\phi_2 \sin\Delta\lambda}{\sin A}$ +.05029163 $\sin B$ +1.000 00000 B 90 00 11.342

d 2 52 57.750

$K = (\sin\phi_1 - \sin\phi_2)^2$ $+1.41622 \times 10^{-6}$ $H = (d + 3 \sin d)/(1 - \cos d)$ +158.988826

$L = (\sin\phi_1 + \sin\phi_2)^2$ +3.527 61717 $G = (d - 3 \sin d)/(1 + \cos d)$ −.050 294892

$\delta d = -(f/4)(HK + GL)$ +.00015 01777 $s = a(d + \delta d)$ 321,862.977 meters

d (radians) +.0503 12752 s 173.7921 n.m.

$d + \delta d$ (rad) +.0504 62929 $T = d/\sin d$ 1.000 42

2A 164° 15′ 35.″154 2B 180° 00′ 22.″684

$\sin 2A$ +.271 27641 $\sin 2B$ −.000 10998

$U = (f/2) \cos^2\phi_1 \sin 2A$ $+5.48169 \times 10^{-5}$ $V = (f/2) \cos^2\phi_2 \sin 2B$ -2.181×10^{-8}

VT -2.182×10^{-8} UT $+5.4840 \times 10^{-5}$

$\delta A = VT - U$ -5.4839×10^{-5} $\delta B = -UT + V$ -5.4862×10^{-5}

$+\delta A$ − 0° 0′ 11.″311 $+\delta B$ − 0° 0′ 11.″316

$-A$ − 82 07 47.577 $+B$ + 90 00 11.342

+ 180 + 180

α_{1-2} 97° 52′ 01.″112 α_{2-1} 270° 00′ 00.″026

$\alpha_{1-2} = \alpha_{AB} = 180° - A + \delta A$ $\alpha_{2-1} = \alpha_{BA} = 180° + B + \delta B$

Line No. 3 (See Tables 1,2 - pages 65,66)

COMPUTING FORM, ANDOYER-LAMBERT

(No conversion to parametric latitudes)

Clarke Spheroid, 1866 a = 6,378,206.4 meters

f/2 = 0.00169503765, f/4 = 0.000847518825

1 radian = 206,264.8062 seconds

ϕ_1 13° 04′ 12.564″ 1. Origin	λ_1 14° 51′ 13.283″
ϕ_2 10 00 00.000 2. Terminus	λ_2 18 00 00.000
$\sin\phi_2$ .17364818 2. West of 1.	$\Delta\lambda = \lambda_2 - \lambda_1$ 3 08 46.717
$\cos\phi_2$ .98480775 $\sin\phi_1$ .22614397	$\sin\Delta\lambda$ .05488588
$\cos^2\phi_2$ .96984630 $\cos\phi_1$ .97409389	$\cos\Delta\lambda$ .99849263
$\cos^2\phi_1$ .94885891	$\cos d = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos\Delta\lambda$ .99711869
$K = (\sin\phi_1 - \sin\phi_2)^2$ +.00275581	d 4° 21′ 01.″722
$L = (\sin\phi_1 + \sin\phi_2)^2$ .15983376	d (radians) .075930171
$H = (d + 3\sin d)/(1 - \cos d)$ +105.33468	sin d .07585723
$G = (d - 3\sin d)/(1 + \cos d)$ −.07593015	$s = a(d + \delta d)$ 482,794.743 meters
$\delta d = -f(HK + GL)/4$ -2.35734×10^{-4}	s 260.6883 n.m.
$R = \sin\Delta\lambda / \sin d$ .72354184	$T = d/\sin d$ 1.0009616
$\sin A = R\cos\phi_2$ .71254961	$\sin B = R\cos\phi_1$ .70479769
A 134° 33′ 26.″138	B 44° 48′ 47.″526
2A 269 06 52.276	2B 89 37 35.052
sin 2A −.99988058	sin 2B +.99997874
$U = (f/2)\cos^2\phi_1 \sin 2A$	$V = (f/2)\cos^2\phi_2 \sin 2B$
U (rad) −.0016081595	V (rad) +.0016443891
U	V
VT + .0016454718	UT −.001609706
$\delta A = VT - U$ + ° +11′ 11.″110	$\delta B = -UT + V$ + ° 11′ 11.″103
$\alpha_{AB} = 180° - A + \delta A$ 45° 37′ 44.″972	$\alpha_{BA} = 180° + B + \delta B$ 224° 59′ 58.″629

Line No. 4 (See Tables 1,2 - pages 65,66)

COMPUTING FORM, ANDOYER-LAMBERT

(No conversion to parametric latitudes)

Clarke Spheroid, 1866 a = 6,378,206.4 meters

f/2 = 0.00169503765, f/4 = 0.000847518825

1 radian = 206,264.8062 seconds

ϕ_1 73° 35′ 09.206″	1. Origin	λ_1 3° 26′ 35.101″
ϕ_2 70 00 00.000	2. Terminus	λ_2 18 00 00.000
$\sin\phi_2$ .93969262	2. West of 1.	$\Delta\lambda = \lambda_2 - \lambda_1$ 14 33 24.899
$\cos\phi_2$ .34202014	$\sin\phi_1$ .95924441	$\sin\Delta\lambda$ .25134162
$\cos^2\phi_2$ .11697778	$\cos\phi_1$ .28257768	$\cos\Delta\lambda$ .96789844
$\cos^2\phi_1$ .07985015	$\cos d = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos\Delta\lambda$ .99493962	

$K = (\sin\phi_1 - \sin\phi_2)^2$ .000382272	d 5° 45′ 59.″408
$L = (\sin\phi_1 + \sin\phi_2)^2$ 3.60596184	d (radians) .10064445
$H = (d + 3\sin d)/(1 - \cos d)$ +79.4541793	sin d .10047463
$G = (d - 3\sin d)/(1 + \cos d)$ −.100644369	$s = a(d + \delta d)$ 643,728.709 meters
$\delta d = -f(HK + GL)/4$ +.00028184	s 347.5857 n.m.
$R = \sin\Delta\lambda / \sin d$ 2.501543125	$T = d/\sin d$ 1.0016902
$\sin A = R\cos\phi_2$ .85557813	$\sin B = R\cos\phi_1$ .70688025
A 121° 10′ 34.″813	B 44° 58′ 53.″930
2A 242 21 09.626	2B 89 57 47.860
sin 2A −.88582060	sin 2B +.99999980
$U = (f/2)\cos^2\phi_1 \sin 2A$	$V = (f/2)\cos^2\phi_2 \sin 2B$
U (rad) -1.19895×10^{-4}	V (rad) $+1.98282 \times 10^{-4}$
U	V
VT $+1.98617 \times 10^{-4}$	UT -1.20098×10^{-4}
$\delta A = VT - U$ + 0° 01′ 05.″698	$\delta B = -UT + V$ + 0° 01′ 05.″671
$\alpha_{AB} = 180° - A + \delta A$ 58° 50′ 30.″335	$\alpha_{BA} = 180° + B + \delta B$ 224° 59′ 59.″601

Line No. 5 (See Tables 1,2 - pages 65,66)

DISTANCE COMPUTING FORM, ANDOYER-LAMBERT APPROXIMATION

(No conversion to parametric latitudes)

Clarke Spheroid 1866 $a = 6{,}378{,}206.4$ meters

$f/2 = 0.00169503765$, $f/4 = 0.000847518825$

1 radian = 206,264.8062 seconds

ϕ_1 39° 37′ 06.″613 1. Origin λ_1 8° 36′ 43.″276

ϕ_2 40° 00′ 00.″000 2. Terminus λ_2 18° 00′ 00.″000

$\sin\phi_1$ .637 67279 2. West of 1. $\Delta\lambda = \lambda_2 - \lambda_1 =$ 9 23 16.734

$\cos\phi_1$ .770 30 735 $\sin\phi_2$ .642 78761 $\sin\Delta\lambda$ .163 11897

$\tan\phi_1$ .827 81605 $\cos\phi_2$ .766 04444 $\cos\Delta\lambda$ .986 60641

$\tan\phi_2$ .839 09963 $\cos d = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos\Delta\lambda$ +.992 07441

$M = \cos\phi_1 \tan\phi_2 - \sin\phi_1 \cos\Delta\lambda$ +.017 23255 $\cot A = \dfrac{M}{\sin\Delta\lambda}$ +.105 64406

$N = \cos\phi_2 \tan\phi_1 - \sin\phi_2 \cos\Delta\lambda$ −.00003450 $\cot B = \dfrac{N}{\sin\Delta\lambda}$ −.000 21150

$\sin d = \dfrac{\cos\phi_1 \sin\Delta\lambda}{\sin B}$ .12565174 $\sin A$ .0994 46595 A 83° 58′ 09.″874

$= \dfrac{\cos\phi_2 \sin\Delta\lambda}{\sin A}$ .12565174 $\sin B$ .99999998 B 90° 00′ 43.″625

$K = (\sin\phi_1 - \sin\phi_2)^2$ +2.6161384 × 10⁻⁵ $H = (d + 3\sin d)/(1 - \cos d)$ 63.4577577

$L = (\sin\phi_1 + \sin\phi_2)^2$ 1.6395788 $G = (d - 3\sin d)/(1 + \cos d)$ −.125 98454

$\delta d = -(f/4)(HK + GL)$ +.00017366 $s = a(d + \delta d)$ 804,664.697 meters

d (radians) +.125 98490 s 434.4842 n.m.

$d + \delta d$ (rad) +.126 15846 $T = d/\sin d$ 1.002 65066

2A 167° 56′ 19.″748 2B 180° 01′ 27.″250

sin 2A +.208 95605 sin 2B −.000 42300

$U = (f/2)\cos^2\phi_1 \sin 2A$ +.0002 10166 $V = (f/2)\cos^2\phi_2 \sin 2B$ −4.21 × 10⁻⁷

VT −4.22 × 10⁻⁷ UT +.000210 723

$\delta A = VT - U$ −.0002 10588 $\delta B = -UT + V$ −.0002 11144

$+\delta A$ − 0° ′ 43.″437 $+\delta B$ − 0° ′ −43.″552

$-A$ − 83 58 09.874 $+B$ + 90 00 43.625

+ 180 + 180

α_{1-2} 96° 01′ 06.″689 α_{2-1} 270° 00′ 00.″073

$\alpha_{1-2} = \alpha_{AB} = 180° - A + \delta A$ $\alpha_{2-1} = \alpha_{BA} = 180° + B + \delta B$

Line No. 6 (See Tables 1,2 - pages 65,66)

DISTANCE COMPUTING FORM, ANDOYER-LAMBERT APPROXIMATION

(No conversion to parametric latitudes)

Clarke Spheroid 1866 a = 6,378,206.4 meters

f/2 = 0.00169503765, f/4 = 0.000847518825

1 radian = 206,264.8062 seconds

ϕ_1 44° 54′ 28.507″ 1. Origin λ_1 10° 47′ 43.883″

ϕ_2 40 00 00.000 2. Terminus λ_2 18 00 00.000

$\sin\phi_1$ 0.705 96946 2. West of 1. $\Delta\lambda = \lambda_2 - \lambda_1 =$ 7 12 16.117

$\cos\phi_1$ 0.708 24228 $\sin\phi_2$ .642 78761 $\sin\Delta\lambda$ 0.125 41075

$\tan\phi_1$ 0.996 79091 $\cos\phi_2$ .766 04444 $\cos\Delta\lambda$ 0.992 10491

$\tan\phi_2$ .8390996 3 $\cos d = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos\Delta\lambda$ 0.992 05004

$M = \cos\phi_1 \tan\phi_2 - \sin\phi_1 \cos\Delta\lambda$ −.106 10993 $\cot A = \dfrac{M}{\sin\Delta\lambda}$ −.846 09916

$N = \cos\phi_2 \tan\phi_1 - \sin\phi_2 \cos\Delta\lambda$ +.125 87339 $\cot B = \dfrac{N}{\sin\Delta\lambda}$ +1.00368 900

$\sin d = \dfrac{\cos\phi_1 \sin\Delta\lambda}{\sin B}$ .125 84404 $\sin A$ .763 40687 A 130° 14′ 04.316″

$= \dfrac{\cos\phi_2 \sin\Delta\lambda}{\sin A}$ .125 84404 $\sin B$ .705 80373 B 44 53 40.246

d 7 13 46.202

$K = (\sin\phi_1 - \sin\phi_2)^2$ 3.99 1946 × 10^{-3} $H = (d + 3\sin d)/(1 - \cos d)$ +63.3601565

$L = (\sin\phi_1 + \sin\phi_2)^2$ 1.819 14563 $G = (d - 3\sin d)/(1 + \cos d)$ −.126178331

$\delta d = -(f/4)(HK + GL)$ −.000 019826 $s = a(d + \delta d)$ 804,666.623 meters

d (radians) .126 178588 s 434.4852 n.m.

$d + \delta d$ (rad) .126 158762 $T = d/\sin d$ 1.002 658433

2A 260° 28′ 08.632″ 2B 89° 47′ 20.492″

sin 2A −.986 19633 sin 2B .999 99322

$U = (f/2)\cos^2\phi_1 \sin 2A$ −8.385065 × 10^{-4} $V = (f/2)\cos^2\phi_2 \sin 2B$ +9.946832 × 10^{-4}

VT +9.973265 × 10^{-4} UT −8.407356 × 10^{-4}

$\delta A = VT - U$ +18.35833 × 10^{-4} $\delta B = -UT + V$ −18.354178 × 10^{-4}

$+\delta A$ + 0° 6′ 18.668″ $+\delta B$ + 0° 6′ 18.582″

$-A$ −130 14 04.316 $+B$ 44 53 40.246

+180 +180

α_{1-2} 49° 52′ 14.352″ α_{2-1} 224° 59′ 58.828″

$\alpha_{1-2} = \alpha_{AB} = 180° - A + \delta A$ $\alpha_{2-1} = \alpha_{BA} = 180° + B + \delta B$

Line No. 7 (See Tables 1,2 - pages 65,66)

DISTANCE COMPUTING FORM, ANDOYER-LAMBERT APPROXIMATION

(No conversion to parametric latitudes)

Clarke Spheroid 1866 a = 6,378,206.4 meters

f/2 = 0.00169503765, f/4 = 0.000847518825

1 radian = 206,264.8062 seconds

ϕ_1 +76° 00' 26.603"N 1. Origin λ_1 28° 42' 03.567" E

ϕ_2 +70° 00' 00.000"N 2. Terminus λ_2 18° 00' 00.000" W

$\sin\phi_1$ .970 326 92 2. West of 1. $\Delta\lambda = \lambda_2 - \lambda_1 =$ 46 42 03.567

$\cos\phi_1$ .241 796 75 $\sin\phi_2$ .939 692 62 $\sin\Delta\lambda$ .727 784 62

$\tan\phi_1$ 4.0129858 $\cos\phi_2$ .342 020 14 $\cos\Delta\lambda$ .685 805 77

$\tan\phi_2$ 2.7474 7742 $\cos d = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos\Delta\lambda$ .968 523 475

$M = \cos\phi_1 \tan\phi_2 - \sin\phi_1 \cos\Delta\lambda$ −.00 112469 $\cot A = \frac{M}{\sin\Delta\lambda}$ −.0015 4536

$N = \cos\phi_2 \tan\phi_1 - \sin\phi_2 \cos\Delta\lambda$ +.72807535 $\cot B = \frac{N}{\sin\Delta\lambda}$ +1.0003 9947

$\sin d = \frac{\cos\phi_1 \sin\Delta\lambda}{\sin B}$ .24891730 $\sin A$ .99999880 A 90° 05' 18".753

$= \frac{\cos\phi_2 \sin\Delta\lambda}{\sin A}$ .24891730 $\sin B$ .706 96 556 B 44 59 18.810

d 14 24 48.430

$K = (\sin\phi_1 - \sin\phi_2)^2$ 9.3846034×10^{-4} $H = (d + 3\sin d)/(1 - \cos d)$ 31.717 4323

$L = (\sin\phi_1 + \sin\phi_2)^2$ 3.648 17464 $G = (d - 3\sin d)/(1 + \cos d)$ −.25155 3703

$\delta d = -(f/4)(HK + GL)$ +.00075 25512 $s = a(d + \delta d)$ 1,609,315.609 meters

d (radians) .25156 22076 s 868.9608 n.m.

$d + \delta d$ (rad) .25231 47588 $T = d/\sin d$ 1.01062 5647

2A 180° 10' 37".506 2B 89° 58' 37".620

$\sin 2A$ −.0030 9071 $\sin 2B$ +.9999999 2

$U = (f/2)\cos^2\phi_1 \sin 2A$ $-3.0629403 \times 10^{-7}$ $V = (f/2)\cos^2\phi_2 \sin 2B$ $+1.98281725 \times 10^{-4}$

VT $+2.0038860 \times 10^{-4}$ UT $-3.0954860 \times 10^{-7}$

$\delta A = VT - U$ $+2.0069489 \times 10^{-4}$ $\delta B = -UT + V$ $+1.9889232 \times 10^{-4}$

$+\delta A$ + 41.396 $+\delta B$ + 41.024

$-A$ − 90 05 18.753 $+B$ +44 59 18.810

+ 180 + 180

α_{1-2} 89° 55' 22".643 α_{2-1} 224° 59' 59".834

$\alpha_{1-2} = \alpha_{AB} = 180° - A + \delta A$ $\alpha_{2-1} = \alpha_{BA} = 180° + B + \delta B$

Line No. 8 (See Tables 1,2 - pages 65,66)

COMPUTING FORM, ANDOYER-LAMBERT

(No conversion to parametric latitudes)

Clarke Spheroid, 1866 $a = 6{,}378{,}206.4$ meters

$f/2 = 0.00169503765$, $f/4 = 0.000847518825$

1 radian = 206,264.8062 seconds

ϕ_1 27° 49′ 42.″130N 1. Origin	λ_1 32° 54′ 12.″997E
ϕ_2 40 00 00.000 2. Terminus	λ_2 18 00 00.000W
$\sin \phi_2$ .64278761 2. West of 1.	$\Delta\lambda = \lambda_2 - \lambda_1$ 50 54 12.997
$\cos \phi_2$ .76604444 $\sin \phi_1$ .46682458	$\sin \Delta\lambda$ .77608614
$\cos^2\phi_2$ .58682408 $\cos \phi_1$ .88434994	$\cos \Delta\lambda$ .63062691
$\cos^2\phi_1$ .78207482	$\cos d = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos\Delta\lambda$ .72728811
$K = (\sin\phi_1 - \sin\phi_2)^2$ .030962988	d 43° 20′ 25.″706
$L = (\sin\phi_1 + \sin\phi_2)^2$ 1.23123921	d (radians) .756433968
$H = (d + 3\sin d)/(1 - \cos d)$ +10.3238286	sin d .68633228
$G = (d - 3\sin d)/(1 + \cos d)$ −.754108629	$s = a(d + \delta d)$ 4,827,983.105 meters
$\delta d = -f(HK + GL)/4$ +.000515996	s 2606.9023 n.m.
$R = \sin\Delta\lambda / \sin d$ 1.130773187	$T = d/\sin d$ 1.102139575
$\sin A = R\cos\phi_2$ .86622251	$\sin B = R\cos\phi_1$ .99999920
A 60° 01′ 21.″339	B 90° 04′ 21.″000
2A 120 02 42.678	2B 180 08 42.000
sin 2A .86563079	sin 2B −.002 53072
$U = (f/2)\cos^2\phi_1 \sin 2A$	$V = (f/2)\cos^2\phi_2 \sin 2B$
U (rad) .00114752022	V (rad) -2.517279×10^{-6}
U 3′ 56.″693	V − 00.″519
VT − 00.″572	UT 4′ 20.″869
$\delta A = VT - U$ − 3′ 57.″265	$\delta B = -UT + V$ − 4′ 21.″388
$\alpha_{AB} = 180° - A + \delta A$ 119° 54′ 41.″396	$\alpha_{BA} = 180° + B + \delta B$ 269° 59′ 57.″612

Line No. 9 (See Tables 1,2 - pages 65,66)

COMPUTING FORM, ANDOYER-LAMBERT

(No conversion to parametric latitudes)

Clarke Spheroid, 1866 a = 6,378,206.4 meters

f/2 = 0.00169503765, f/4 = 0.000847518825

1 radian = 206,264.8062 seconds

ϕ_1 35° 18′ 45.644″N 1. Origin λ_1 102° 02′ 29.310E

ϕ_2 40 00 00.000N 2. Terminus λ_2 18 00 00.000W

$\sin\phi_2$ .64278761 2. West of 1. $\Delta\lambda = \lambda_2 - \lambda_1$ 120 02 29.310

$\cos\phi_2$ .76604444 $\sin\phi_1$ .57803821 $\sin\Delta\lambda$ 0.86566309

$\cos^2\phi_2$ .58682408 $\cos\phi_1$ .81600970 $\cos\Delta\lambda$ 0.50062701

$\cos^2\phi_1$ .66587183 $\cos d = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos\Delta\lambda$ .05861401

$K = (\sin\phi_1 - \sin\phi_2)^2$ .004192 4848 d 86° 38′ 23.″060

$L = (\sin\phi_1 + \sin\phi_2)^2$ 1.49041568 d (radians) 1.51214871

$H = (d + 3\sin d)/(1 - \cos d)$ 4.78761188 sin d 0.99828068

$G = (d - 3\sin d)/(1 + \cos d)$ −1.40059863 $s = a(d + \delta d)$ 9,655,972.218 meters

$\delta d = -f(HK + GL)/4$ +.00175216 s 5213.8079 n.m.

$R = \sin\Delta\lambda / \sin d$ +.867154005 $T = d/\sin d$ 1.51475305

$\sin A = R\cos\phi_2$ +.66427850 $\sin B = R\cos\phi_1$ +.70760611

A 41° 37′ 37.″191 B 45° 02′ 25.″708

2A 83 15 14.382 2B 90 04 51.416

sin 2A +.99307665 sin 2B +.99999900

$U = (f/2)\cos^2\phi_1 \sin 2A$ $V = (f/2)\cos^2\phi_2 \sin 2B$

U (rad) .00112 0864 V (rad) .0009946879

U ° 3′ 51.″195 V ° 3′ 25.″169

VT ° 5′ 10.″780 UT ° 5′ 50.″203

$\delta A = VT - U$ ° 1′ 19.″585 $\delta B = -UT + V$ ° −2′ 25.″034

$\alpha_{AB} = 180° - A + \delta A$ 138° 23′ 42.″394 $\alpha_{BA} = 180° + B + \delta B$ 225° 00′ 00.″674

Line No. 10 (See Tables 1,2 - pages 65,66)

INVERSE COMPUTATION
(Andoyer-Lambert Formula)
Clarke 1866 Ellipsoid
40-50-6000 Line

ϕ_1 40° 00' 00".000N ϕ_2 35 18 45.644N	1. Point of Origin 2. Terminal Point Point 1 should be west of point 2	λ_1 18° 00' 00".000W λ_2 102 02 29.370E $\Delta\lambda$ 120° 02' 29".370

$\tan\beta = b/a \tan\phi$ $\tan\phi_1$ 0.83909963 $\tan\phi_2$ 0.70837174	$\sin\Delta\lambda$ 0.86566309 $\cos\Delta\lambda$ −0.50062701

	tan	angle	sin	cos
β_1	0.83625502	39° 54' 15".203	0.64150618	0.76711787
β_2	0.70597031	35 13 15.443	0.57673115	0.81693401

$$\cot A = \frac{\cos\beta_1 \tan\beta_2 - \sin\beta_1 \cos\Delta\lambda}{\sin\Delta\lambda} \qquad \cot B = \frac{\cos\beta_2 \tan\beta_1 - \sin\beta_2 \cos\Delta\lambda}{\sin\Delta\lambda}$$

	cot	angle	sin	cos (5 places)
A	0.99659760	45° 05' 51".495	0.70831073	0.705901
	tan B			
B	0.89069853	41 41 29.068	0.66511838	0.746738

$$\sin\sigma = \frac{\cos\beta_1 \sin\Delta\lambda}{\sin B} = \frac{\cos\beta_2 \sin\Delta\lambda}{\sin A}$$

$$\cos\sigma = \sin\beta_1 \sin\beta_2 + \cos\beta_1 \cos\beta_2 \cos\Delta\lambda$$

$\sin\sigma$ 0.99841720
$\cos\sigma$ 0.05624132
σ 86° 46' 33".271
σ'' 312393.271
σ 1.51452532 radians

$$M = (\sin\beta_1 + \sin\beta_2)^2 \qquad N = (\sin\beta_1 - \sin\beta_2)^2$$

$$U = \frac{\sigma - \sin\sigma}{1 + \cos\sigma} \qquad V = \frac{\sigma + \sin\sigma}{1 - \cos\sigma}$$

M 1.48410219
U 0.48862709
N 0.00419580
V 2.66269606

$\dfrac{f\sigma''}{\sin\sigma}$ 1060.7155

$s = a\sigma - H(MU + NV)$
$a\sigma$ 9659955.089
$-H(MU + NV)$ − 3980.422

s 9 655 974 .667 meters

$$\delta A'' = -\cos^2\beta_2 \sin B \cos B\left(\frac{f\sigma''}{\sin\sigma}\right) \qquad \delta A'' \quad -351.593$$

$$\delta B'' = -\cos^2\beta_1 \sin A \cos A\left(\frac{f\sigma''}{\sin\sigma}\right) \qquad \delta B'' \quad -312.098$$

A	45°	05'	51".495	B	41°	41'	29".068
δA −		05	51.593	δB −		5	12.098
A_f	44	59	59.902	B_f	41	36	16.970

$\alpha_1 = 180° + A_f$ 224° 59' 59".902 | $\alpha_2 = 180° - B_f$ 138° 23' 43".030

Line No. 10 as computed by ACIC, converting to parametric latitude.

(From Page 39 of the ACIC Technical Report No. 80 – August 1957)

COMPUTING FORM, ANDOYER-LAMBERT

(No conversion to parametric latitudes)

Clarke Spheroid, 1866 a = 6,378,206.4 meters

f/2 = 0.00169503765, f/4 = 0.000847518825

1 radian = 206,264.8062 seconds

ϕ_1 18° 29′ 57.900″ 1. Origin λ_1 67° 07′ 30.300″

ϕ_2 43° 03′ 19.600″ 2. Terminus λ_2 115° 52′ 54.700″

$\sin \phi_2$ .682 70576 2. West of 1. $\Delta\lambda = \lambda_2 - \lambda_1$ 48 45 24.400

$\cos \phi_2$ .730 69339 $\sin \phi_1$ .317 29500 $\sin \Delta\lambda$ .751 91780

$\cos^2 \phi_2$ .533 91283 $\cos \phi_1$ .948 32688 $\cos \Delta\lambda$ .659 25687

$\cos^2 \phi_1$ .899 32387 $\cos d = \sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2 \cos \Delta\lambda$ .673 44206

$K = (\sin \phi_1 - \sin \phi_2)^2$ .133 52502 d 47 40 00.179

$L = (\sin \phi_1 + \sin \phi_2)^2$ 1.00000152 d (radians) .831 941144

$H = (d + 3 \sin d)/(1 - \cos d)$ +9.338 80575 sin d .739 24001

$G = (d - 3 \sin d)/(1 + \cos d)$ −.828 100908 $s = a(d + \delta d)$ 5,304028.110 meters

$\delta d = -f(HK + GL)/4$ $-3.5499347 \times 10^{-4}$ s 2863.9461 n.m.

$R = \sin \Delta\lambda / \sin d$ 1.017149761 $T = d/\sin d$ 1.125 40059

$\sin A = R \cos \phi_2$ .943 23461 $\sin B = R \cos \phi_1$ .964 59046

A 48 00 24.496 B 105 17 34.164

2A 96 00 48.992 2B 210 35 08.328

sin 2A .994 49704 sin 2B −.508 82577

$U = (f/2) \cos^2 \phi_1 \sin 2A$ $V = (f/2) \cos^2 \phi_2 \sin 2B$

U (rad) $1.515\ 9992 \times 10^{-3}$ V (rad) $-4.6048852 \times 10^{-4}$

U V

VT -5.18234×10^{-4} UT ' 1.706106×10^{-3}

$\delta A = VT - U$ − 6 59.591 $\delta B = -UT + V$ − 7 26.892

$\alpha_{AB} = 180° - A + \delta A$ 131° 52′ 35.″913 $\alpha_{BA} = 180° + B + \delta B$ 285° 10′ 07.″272

Line No. 11 (See Tables 1,2 - pages 65,66)

DISTANCE COMPUTING FORM, ANDOYER-LAMBERT APPROXIMATION

(No conversion to parametric latitudes)

Clarke Spheroid 1866 a = 6,378,206.4 meters

f/2 = 0.00169503765, f/4 = 0.000847518825

1 radian = 206,264.8062 seconds

ϕ_1 55° 45′ 19.5″(N) 1. Moscow — λ_1 −37° 34′ 15.″450 (E)

ϕ_2 −33 56 03.5 (S) 2. Cape of Good Hope — λ_2 −18 28 41.400 (E)

sin ϕ_1 +.826 64295 — 2. West of 1. — $\Delta\lambda = \lambda_2 - \lambda_1$ = +19 05 34.050

cos ϕ_1 +.56272678 — sin ϕ_2 −.558 24198 — sin $\Delta\lambda$ +.327 09901

tan ϕ_1 1.46899522 — cos ϕ_2 +.82967819 — cos $\Delta\lambda$ +.94499007

tan ϕ_2 −.67284157 — $\cos d = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos\Delta\lambda$ −.62026782

$M = \cos\phi_1 \tan\phi_2 - \sin\phi_1 \cos\Delta\lambda$ −1.159 79535 — $\cot A = \dfrac{M}{\sin\Delta\lambda}$ −3.54570119

$N = \cos\phi_2 \tan\phi_1 - \sin\phi_2 \cos\Delta\lambda$ +1.74632643 — $\cot B = \dfrac{N}{\sin\Delta\lambda}$ +5.33883129

$\sin d = \dfrac{\cos\phi_1 \sin\Delta\lambda}{\sin B}$ .99979459 — sin A +.27144267 — A 164 14 59.524

$= \dfrac{\cos\phi_2 \sin\Delta\lambda}{\sin A}$ .99979459 — sin B +.184 10519 — B 10 36 32.283

d 91 09 40.825

$K = (\sin\phi_1 - \sin\phi_2)^2$ +1.91790627 — $H = (d + 3\sin d)/(1 - \cos d)$ +4.499259135

$L = (\sin\phi_1 + \sin\phi_2)^2$ +.072039081 — $G = (d - 3\sin d)/(1 + \cos d)$ −1.43745225

$\delta d = -(f/4)(HK + GL)$ −.007225610 — $s = a(d + \delta d)$ 10,102,057.965 meters

d (radians) +1.591065538 — s 5454.6749 n.m.

$d + \delta d$ (rad) +1.583839928 — $T = d/\sin d$ 1.59139242

2A 328° 29′ 59.″048 — 2B 21° 13′ 04.″566

sin 2A −.522 50250 — sin 2B +.361 91639

$U = (f/2)\cos^2\phi_1 \sin 2A$ −2.804548×10^{-4} — $V = (f/2)\cos^2\phi_2 \sin 2B$ +4.2228626×10^{-4}

VT +6.7202315×10^{-4} — UT −4.4631364×10^{-4}

$\delta A = VT - U$ +9.52478×10^{-4} — $\delta B = -UT + V$ +8.685 999×10^{-4}

$+\delta A$ + 3′ 16.″463 — $+\delta B$ + 2′ 59.″162

$-A$ −164 14 59.524 — $+B$ +10 36 32.283

+180 — +180

α_{1-2} 15° 48′ 16.″939 — α_{2-1} 190° 39′ 31.″445

$\alpha_{1-2} = \alpha_{AB} = 180° - A + \delta A$ — $\alpha_{2-1} = \alpha_{BA} = 180° + B + \delta B$

Line No. 12 (See Tables 1,2 - pages 65,66)

APPENDIX 3

Computations

Using Forsyth-Andoyer-Lambert Type

Second Order Formulae

Without Conversion to Parametric Latitude

DISTANCE COMPUTING FORM — ANDOYER-LAMBERT
TYPE APPROXIMATION WITH SECOND ORDER TERMS
(No conversion to parametric latitudes)
Clarke Spheroid 1866, a = 6,378,206.4 meters
$f/2 = 0.00169503765$, $f/4 = 0.000847518825$, $f^2/128 = 0.0897860195 \times 10^{-6}$

ϕ_1 40° 30′ 37.757″ 1. Origin λ_1 17° 19′ 43.280″
ϕ_2 40 00 00.000 2. Terminus λ_2 18 00 00.000
$\sin\phi_1$ +.649 58723 2. west of 1. $\Delta\lambda = \lambda_2 - \lambda_1$ 40′ 16.720
$\cos\phi_1$ +.760 28707 $\sin\phi_2$ +.642 78761 $\sin\Delta\lambda$ +.011 71632
$\tan\phi_1$ +.854 39731 $\cos\phi_2$ +.766 04444 $\cos\Delta\lambda$ +.999 93136
$\tan\phi_2$ +.839 09963 $\cos d = \sin\phi_1\sin\phi_2 + \cos\phi_1\cos\phi_2\cos\Delta\lambda$ +.999 92033
$M = \cos\phi_1\tan\phi_2 - \sin\phi_1\cos\Delta\lambda$ −.011 58604 $\cot u = M/\sin\Delta\lambda$ −.988 88047
$N = \cos\phi_2\tan\phi_1 - \sin\phi_2\cos\Delta\lambda$ +.011 76282 $\cot v = N/\sin\Delta\lambda$ +1.00396882
$\sin d = \cos\phi_1\sin\Delta\lambda/\sin v = \cos\phi_2\sin\Delta\lambda/\sin u$ +.01262251 u 134 40 46.816
$\csc d$ +79.22235458 $\cot d$ +79.21723341 v 44 53 11.497
$1+\cos d$ +1.99992033 $1-\cos d$ +.00007967 $\sin u$ +.711 04900
$(\sin\phi_1+\sin\phi_2)^2$ +1.67023273 $(\sin\phi_1-\sin\phi_2)^2$ +4.62348321 × 10⁻⁵ $\sin v$ +.705 70498
$K_1=(\sin\phi_1+\sin\phi_2)^2/(1+\cos d)$ +.83514963 $K_2=(\sin\phi_1-\sin\phi_2)^2/(1-\cos d)$ +.58032925 9
$X=K_1+K_2$ +1.4154 78892 $Y=K_1-K_2$ +.25482037 4 XY +.36069286 1
X^2 +2.00358 0494 Y^2 +.064933423 d_r +.01262293382 d_r^2 +.00015933846
$A=64d_r+16d_r^2\cot d$ +.8280635278 $D=48\sin d+8d_r^2\csc d$ +.615 97917
$B=-2D$ −1.23195834 $E=30\sin 2d$ +.75729030 $\sin 2d$ +.025 24301
$C=-(30d_r+8d_r^2\cot d+E/2)$ −.7674310463 AX +1.172 106445
BY −.313928085 CX^2 −1.537609875 DXY +.2221792891
EY^2 +.049 173 4514 $\Sigma = AX+BY+CX^2+DXY+EY^2$ −.40807 8774
$\delta d_f = -(f/4)(Xd_r - 3Y\sin d)$ -6.96498×10^{-6} $\delta d_f^2 = +(f^2/128)\Sigma$ -3.66398×10^{-8}
$d_r + \delta d_f$ .012 61596884 $d_r + \delta d_f + \delta d_f^2$ .012 615 93220
$S(\delta d_f) = a(d_r + \delta d_f)$ 80,467.253 m $S(\delta d_{f^2}) = a(d_r + \delta d_f + \delta d_{f^2})$ 80,467.020 m

$T = d/\sin d$ 1.00003 3576
2u 269° 21′ 33.″632 2v 89° 46′ 22.″994
$\sin 2u$ −.999 93749 $\sin 2v$ +.999 99216
$U=(f/2)\cos^2\phi_1\sin 2u$ $-9.79732265 \times 10^{-4}$ $V=(f/2)\cos^2\phi_2\sin 2v$ $+9.9468 1111 \times 10^{-4}$
VT $+9.947145 \times 10^{-4}$ UT $-9.797 6516 \times 10^{-4}$
$\delta u = VT - U$ +.0019744468 $\delta v = -UT + V$ +.0019744627
$+\delta u$ + 6 47.259 $+\delta v$ + 6 47.262
$-u$ 134 40 46.816 $+v$ 44 53 11.497
+180 ° ′ ″ +180 ° ′ ″
α_{1-2} 45° 26′ 00.″443 α_{2-1} 224° 59′ 58.″759

$\alpha_{1-2} = \alpha_{uv} = 180° - u + \delta u$ $\alpha_{2-1} = \alpha_{vu} = 180° + v + \delta v$

Line No. 1, See Tables 1 and 2. True distance 80,466.490 meters.

DISTANCE COMPUTING FORM — ANDOYER-LAMBERT
TYPE APPROXIMATION WITH SECOND ORDER TERMS

(No conversion to parametric latitudes)

Clarke Spheroid 1866, a = 6,378,206.4 meters

$f/2 = 0.00169503765$, $f/4 = 0.000847518825$, $f^2/128 = 0.0897860195 \times 10^{-6}$

ϕ_1 9° 59′ 48.349″ 1. Origin λ_1 16° 31′ 55.877″

ϕ_2 10 00 00.000 2. Terminus λ_2 18 00 00.000

$\sin\phi_1$ +.173 59355 2. west of 1. $\Delta\lambda = \lambda_2 - \lambda_1$ 1 28 04.123

$\cos\phi_1$ +.98481756 $\sin\phi_2$ +.17364818 $\sin\Delta\lambda$ +.025 61535

$\tan\phi_1$ +.176 26874 $\cos\phi_2$ +.98480775 $\cos\Delta\lambda$ +.999 67188

$\tan\phi_2$ +.176 32698 $\cos d = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos\Delta\lambda$ +.99968177

$M = \cos\phi_1 \tan\phi_2 - \sin\phi_1 \cos\Delta\lambda$ +.000 11432 $\cot u = M/\sin\Delta\lambda$ +.004 46295

$N = \cos\phi_2 \tan\phi_1 - \sin\phi_2 \cos\Delta\lambda$ −.000 00038 $\cot v = N/\sin\Delta\lambda$ −.000 01483

$\sin d = \cos\phi_1 \sin\Delta\lambda/\sin v = \cos\phi_2 \sin\Delta\lambda/\sin u$ +.025 22645 u 89 44 39.457

csc d +39.640 9324 cot d +39.62831 75 v 90 00 03.060

1 + cos d +1.99968177 1 − cos d +.000 31823 sin u +.999 99004

$(\sin\phi_1 + \sin\phi_2)^2$ +.12057612 $(\sin\phi_1 - \sin\phi_2)^2$ 3.1 × 10⁻⁹ sin v +1.000 00000

$K_1 = (\sin\phi_1 + \sin\phi_2)^2/(1 + \cos d)$ +.0602976542 $K_2 = (\sin\phi_1 - \sin\phi_2)^2/(1 - \cos d)$ 9.74138 × 10⁻⁶

$X = K_1 + K_2$ +.0603073956 $Y = K_1 - K_2$ +.0602879128 XY +.0036 3580701

X^2 +.0036 3698196 Y^2 +.0036 3463243 d_r +.0252291222 d_r^2 +.00063650860 7

$A = 64d_r + 16d_r^2 \cot d$ +2.01824 4063 $D = 48 \sin d + 8d_r^2 \csc d$ +1.412 723957

$B = -2D$ −2.825 447914 $E = 30 \sin 2d$ +1.513 1052 sin 2d +.050 43684

$C = -(30d_r + 8d_r^2 \cot d + E/2)$ −1.715 21638 AX +0.121 715 043

BY −.170340357 CX^2 −.00623821 1 DXY +.0051363 9167

EY^2 +.0054995812 $\Sigma = AX + BY + CX^2 + DXY + EY^2$ −.044227552

$\delta d_f = -(f/4)(Xd_r - 3Y \sin d)$ +2.577345 × 10⁻⁶ $\delta d_f^2 = +(f^2/128)\Sigma$ −3.97102 × 10⁻⁹

$d_r + \delta d_f$ +.025 2316 995 $d_r + \delta d_f + \delta d_f^2$ +.025 2316955

$S(\delta d_f) = a(d_r + \delta d_f)$ 160,932.987 m $S(\delta d_{f^2}) = a(d_r + \delta d_f + \delta d_{f^2})$ 160,932.962 m

$T = d/\sin d$ +1.000 105 928

2u 179° 29′ 18.″914 2v 180° 00′ 06.″120

sin 2u +.008 92572 sin 2v −.000 02967

$U = (f/2)\cos^2\phi_1 \sin 2u$ +1.46735 2 × 10⁻⁵ $V = (f/2)\cos^2\phi_2 \sin 2v$ −4.878 × 10⁻⁸

VT −4.878 × 10⁻⁸ UT +1.46751 × 10⁻⁵

$\delta u = VT - U$ −1.4722 × 10⁻⁵ $\delta v = -UT + V$ −1.4724 × 10⁻⁵

$+\delta u$ − 03.037 $+\delta v$ − 03.037

− u 89 44 39.457 + v +90 00 03.060

+180 +180

a_{1-2} 90° 15′ 17.″506 a_{2-1} 270° 00′ 00.″023

$a_{1-2} = a_{uv} = 180° - u + \delta u$ $a_{2-1} = a_{vu} = 180° + v + \delta v$

Line No. 2, See Tables 1 and 2. True distance 160,932.956 meters.

DISTANCE COMPUTING FORM — ANDOYER-LAMBERT
TYPE APPROXIMATION WITH SECOND ORDER TERMS
(No conversion to parametric latitudes)
Clarke Spheroid 1866, a = 6,378,206.4 meters
$f/2 = 0.00169503765$, $f/4 = 0.000847518825$, $f^2/128 = 0.0897860195 \times 10^{-6}$

ϕ_1 69° 48' 05."701 1. Origin λ_1 9° 37' 28."637
ϕ_2 70 00 00.000 2. Terminus λ_2 18 00 00.000
$\sin \phi_1$ .938 50257 2. west of 1. $\Delta\lambda = \lambda_2 - \lambda_1$ 8 22 31.363
$\cos \phi_1$ .345 27226 $\sin \phi_2$ .93969262 $\sin \Delta\lambda$ +.145 65790
$\tan \phi_1$ 2.718 15225 $\cos \phi_2$ .342 02014 $\cos \Delta\lambda$ +.989 33502
$\tan \phi_2$ 2.747 47744 $\cos d = \sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2 \cos \Delta\lambda$ .998 73458
$M = \cos \phi_1 \tan \phi_2 - \sin \phi_1 \cos \Delta\lambda$ +.020 134286 $\cot u = M/\sin \Delta\lambda$ +.138 22996
$N = \cos \phi_2 \tan \phi_1 - \sin \phi_2 \cos \Delta\lambda$ −.000 00 8004 $\cot v = N/\sin \Delta\lambda$ −.000 0549507
$\sin d = \cos \phi_1 \sin \Delta\lambda / \sin v = \cos \phi_2 \sin \Delta\lambda / \sin u$ +.050 29153 u 82° 07' 47."569
$\csc d$ +19.884 02443 $\cot d$ +19.858 86537 v 90 00 11.342
$1 + \cos d$ +1.998 73458 $1 - \cos d$ +.001 26542 $\sin u$ +.99058100
$(\sin \phi_1 + \sin \phi_2)^2$ +3.52761717 $(\sin \phi_1 - \sin \phi_2)^2$ 1.41622×10^{-6} $\sin v$ −1.000 00000
$K_1 = (\sin \phi_1 + \sin \phi_2)^2/(1 + \cos d)$ +1.76492527 $K_2 = (\sin \phi_1 - \sin \phi_2)^2/(1 - \cos d)$ +.001 11917
$X = K_1 + K_2$ +1.7660 4444 $Y = K_1 - K_2$ +1.76380610 XY +3.114 95996
X^2 +3.118 91296 Y^2 +3.111 01196 d_r +.050312752 d_r^2 +.002531373
$A = 64 d_r + 16 d_r^2 \cot d$ +4.024 33915 $D = 48 \sin d + 8 d_r^2 \csc d$ +2.816 66930
$B = -2D$ −5.633 3386 $E = 30 \sin 2d$ +3.0136794 $\sin 2d$ +.100 45598
$C = -(30 d_r + 8 d_r^2 \cot d + E/2)$ −3.418 38377 AX +7.107 16178
BY −9.936 11699 CX^2 −10.6616 4144 DXY +8.773 81209
EY^2 +9.375 59266 $\Sigma = AX + BY + CX^2 + DXY + EY^2$ +4.658 80810
$\delta d_f = -(f/4)(X d_r - 3Y \sin d)$ +.000150177 $\delta d_{f^2} = +(f^2/128)\Sigma$ +.000000 418
$d_r + \delta d_f$ +.0504 62929 $d_r + \delta d_f + \delta d_f^2$ +.0504 63347
$S(\delta d_f) = a(d_r + \delta d_f)$ 321,862,977 m $S(\delta d_{f^2}) = a(d_r + \delta d_f + \delta d_{f^2})$ 321,865.641 m

$T = d/\sin d$ 1.00042

2u 164 15 35.154 2v 180 00 22.684
$\sin 2u$ +.271 27641 $\sin 2v$ −.000 10998
$U = (f/2) \cos^2 \phi_1 \sin 2u$ $+5.48169 \times 10^{-5}$ $V = (f/2) \cos^2 \phi_2 \sin 2v$ -2.181×10^{-8}
VT -2.182×10^{-8} UT $+5.484 \times 10^{-5}$
$\delta u = VT - U$ -5.4839×10^{-5} $\delta v = -UT + V$ -5.4862×10^{-5}
$+\delta u$ − 11.311 $+\delta v$ − 11.316
$-u$ −82 07 47.577 $+v$ +90 00 11.342
+180 ° ' " +180 ° ' "
α_{1-2} 97 52 01.112 α_{2-1} 270 00 00.026

$\alpha_{1-2} = \alpha_{uv} = 180° - u + \delta u$ $\alpha_{2-1} = \alpha_{vu} = 180° + v + \delta v$

Line No. 3, See Tables 1 and 2. True distance 321,866.796 meters.

DISTANCE COMPUTING FORM — ANDOYER-LAMBERT
TYPE APPROXIMATION WITH SECOND ORDER TERMS

(No conversion to parametric latitudes)

Clarke Spheroid 1866, a = 6,378,206.4 meters

$f/2 = 0.00169503765$, $f/4 = 0.000847518825$, $f^2/128 = 0.0897860195 \times 10^{-6}$

	°	′	″			°	′	″
ϕ_1	13	04	12.564	1. Origin	λ_1	14	51	13.283
ϕ_2	10	00	00.000	2. Terminus	λ_2	18	00	00.000

$\sin\phi_1$ +.226 14397 — 2. west of 1. — $\Delta\lambda = \lambda_2 - \lambda_1$ 3 08 46.717

$\cos\phi_1$ +.974 09389 — $\sin\phi_2$ +.17364818 — $\sin\Delta\lambda$ +.054 88588

$\tan\phi_1$ +.232 15829 — $\cos\phi_2$ +.98480775 — $\cos\Delta\lambda$ +.998 49263

$\tan\phi_2$ +.176 32698 — $\cos d = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos\Delta\lambda$ +.997 11869

$M = \cos\phi_1 \tan\phi_2 - \sin\phi_1 \cos\Delta\lambda$ −.054 044 053 — $\cot u = M/\sin\Delta\lambda$ −.98466223

$N = \cos\phi_2 \tan\phi_1 - \sin\phi_2 \cos\Delta\lambda$ +.055 244 855 — $\cot v = N/\sin\Delta\lambda$ +1.00654038

$\sin d = \cos\phi_1 \sin\Delta\lambda/\sin v = \cos\phi_2 \sin\Delta\lambda/\sin u$ +.075 85 7175 — u 134 33 25.986

$\csc d$ +13.1826686 — $\cot d$ +13.1446 8525 — v 44 48 47.676

$1 + \cos d$ +1.997 11869 — $1 - \cos d$ +.00288131 — $\sin u$ +.71255013

$(\sin\phi_1 + \sin\phi_2)^2$ +.15983376 — $(\sin\phi_1 - \sin\phi_2)^2$ +.00275581 — $\sin v$ +.70479821

$K_1 = (\sin\phi_1 + \sin\phi_2)^2/(1 + \cos d)$ +.0800321987 — $K_2 = (\sin\phi_1 - \sin\phi_2)^2/(1 - \cos d)$ +.956443433

$X = K_1 + K_2$ +1.036 475602 — $Y = K_1 - K_2$ −.876411244 — XY −.908 378872

X^2 +1.074281674 — Y^2 +.768096669 — d_r .075930171 — d_r^2 +.005765 3909

$A = 64d_r + 16d_r^2 \cot d$ +6.07207892 — $D = 48 \sin d + 8d_r^2 \csc d$ +4.249 17030

$B = -2D$ −8.49834060 — $E = 30 \sin 2d$ +4.538 31630 — $\sin 2d$ +.15127721

$C = -(30d_r + 8d_r^2 \cot d + E/2)$ −5.15333727 — AX +6.293561654

BY +7.448041257 — CX^2 −5.536 135789 — DXY −3.859856524

EY^2 +3.485865633 — $\Sigma = AX + BY + CX^2 + DXY + EY^2$ +7.831 476231

$\delta d_f = -(f/4)(Xd_r - 3Y \sin d)$ −.00023573398 — $\delta d_f^2 = +(f^2/128)\Sigma$ +7.03157 X 10 −7

$d_r + \delta d_f$ +.075 694 437 — $d_r + \delta d_f + \delta d_f^2$ +.075 695 140

$S(\delta d_f) = a(d_r + \delta d_f)$ 482,794.743 m — $S(\delta d_{f^2}) = a(d_r + \delta d_f + \delta d_{f^2})$ 482,799.226 m

$T = d/\sin d$ +1.00096 2282

	°	′	″		°	′	″
2u	269	06	51.972″	2v	89	37	35.352

$\sin 2u$ −0.998 88056 — $\sin 2v$ +.999 97875

$U = (f/2)\cos^2\phi_1 \sin 2u$ −.0016065511 — $V = (f/2)\cos^2\phi_2 \sin 2v$ +.00164 38911

VT +.0016454730 — UT −.00 16080971

$\delta u = VT - U$ +.003252 0241 — $\delta v = -UT + V$ +.003251 9882

	°	′	″		°	′	″
$+\delta u$		+11	10.778	$+\delta v$		+11	10.771
$-u$	−134	33	25.986	$+v$	+44	48	47.676
	+180				+180		
α_{1-2}	45	37	44.792	α_{2-1}	224	59	58.447

$\alpha_{1-2} = \alpha_{uv} = 180° - u + \delta u$ — $\alpha_{2-1} = \alpha_{vu} = 180° + v + \delta v$

Line No. 4, See Tables 1 and 2. True distance 482,798.163 meters.

DISTANCE COMPUTING FORM — ANDOYER-LAMBERT TYPE APPROXIMATION WITH SECOND ORDER TERMS

(No conversion to parametric latitudes)

Clarke Spheroid 1866, a = 6,378,206.4 meters

$f/2 = 0.00169503765$, $f/4 = 0.000847518825$, $f^2/128 = 0.0897860195 \times 10^{-6}$

ϕ_1 73° 35' 09.206" 1. Origin λ_1 3° 26' 35.101"

ϕ_2 70 00 00.000 2. Terminus λ_2 18 00 00.000

$\sin\phi_1$ .959 24441 2. west of 1. $\Delta\lambda = \lambda_2 - \lambda_1$ 14 33 24.899

$\cos\phi_1$ .282 57768 $\sin\phi_2$ .9396 9262 $\sin\Delta\lambda$ .251 34162

$\tan\phi_1$ +3.394 62200 $\cos\phi_2$ .34202014 $\cos\Delta\lambda$ .967 89844

$\tan\phi_2$ +2.747 47744 $\cos d = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos\Delta\lambda$ +.9949 3963

$M = \cos\phi_1 \tan\phi_2 - \sin\phi_1 \cos\Delta\lambda$ −.152 07537 $\cot u = M/\sin\Delta\lambda$ −.605 05447

$N = \cos\phi_2 \tan\phi_1 - \sin\phi_2 \cos\Delta\lambda$ +.251 50207 $\cot v = N/\sin\Delta\lambda$ +1.00063837

$\sin d = \cos\phi_1 \sin\Delta\lambda/\sin v = \cos\phi_2 \sin\Delta\lambda/\sin u$ +.100 47451 u 121° 10' 34.402"

$\csc d$ +9.952 77310 $\cot d$ +9.902 408382 v 44 58 54.185

$1 + \cos d$ +1.994 93963 $1 - \cos d$ +.005 06037 $\sin u$ +.855 57916

$(\sin\phi_1 + \sin\phi_2)^2$ 3.605 96184 $(\sin\phi_1 - \sin\phi_2)^2$ .000382372 $\sin v$ +.706 88112

$K_1 = (\sin\phi_1 + \sin\phi_2)^2/(1 + \cos d)$ +1.80755437 $K_2 = (\sin\phi_1 - \sin\phi_2)^2/(1 - \cos d)$ +.075 5423022

$X = K_1 + K_2$ +1.8830 9667 $Y = K_1 - K_2$ +1.73201307 XY +3.26154616

X^2 +3.546 05307 Y^2 +2.99986581 d_r +.100644334 d_r^2 +.010129283

$A = 64d_r + 16d_r^2 \cot d$ +8.046 10597 $D = 48 \sin d + 8d_r^2 \csc d$ +5.62929204

$B = -2D$ −11.25858408 $E = 30 \sin 2d$ +5.997 96420 $\sin 2d$ +.19993214

$C = -(30d_r + 8d_r^2 \cot d + E/2)$ −6.3207 4642 AX +15.15159536

BY −19.50000352 CX^2 −24.18672878 DXY +18.36019584

EY^2 +17.99308773 $\Sigma = AX + BY + CX^2 + DXY + EY^2$ +7.81814 663

$\delta d_f = -(f/4)(Xd_r - 3Y \sin d)$ +.0002818391 $\delta d_f^2 = +(f^2/128)\Sigma$ +.000000070196

$d_r + \delta d_f$ +.100 926173 $d_r + \delta d_f + \delta d_f^2$ .100 926875

$S(\delta d_f) = a(d_r + \delta d_f)$ 643,727.963 m $S(\delta d_{f^2}) = a(d_r + \delta d_f + \delta d_{f^2})$ 643,732.440 m

$T = d/\sin d$ 1.001 69022

2u 242° 21' 08.804" 2v 89° 57' 48.370"

$\sin 2u$ −.885 81874 $\sin 2v$ +.999 99980

$U = (f/2)\cos^2\phi_1 \sin 2u$ -1.19895×10^{-4} $V = (f/2)\cos^2\phi_2 \sin 2v$ $+1.98282 \times 10^{-4}$

VT $+1.98617 \times 10^{-4}$ UT -1.20098×10^{-4}

$\delta u = VT - U$ $+3.18512 \times 10^{-4}$ $\delta v = -UT + V$ $+3.18380 \times 10^{-4}$

$+\delta u$ +0° 01' 05.698" $+\delta v$ + 01 05.671

$-u$ −121 10 34.402 $+v$ +44 58 54.185

+180 +180

α_{1-2} 58° 50' 31.296" α_{2-1} 224° 59' 59.856"

$\alpha_{1-2} = \alpha_{uv} = 180° - u + \delta u$ $\alpha_{2-1} = \alpha_{vu} = 180° + v + \delta v$

Line No. 5, See Tables 1 and 2. True distance 643,732.429 meters.

DISTANCE COMPUTING FORM, ANDOYER-LAMBERT
TYPE APPROXIMATION WITH SECOND ORDER TERMS
(No conversion to parametric latitudes)
Clarke Spheroid 1866, $a = 6{,}378{,}206.4$ meters
$f/2 = 0.00169503765$, $f/4 = 0.000847518825$, $f^2/128 = 0.0897860195 \times 10^{-6}$
1 radian = 206,264.8062 seconds

	°	'	"			°	'	"
ϕ_1	9	55	09.138	1. Origin	λ_1	10	39	43.554
ϕ_2	10	0	0	2. Terminus	λ_2	18	0	0
$\phi_m = \frac{1}{2}(\phi_1+\phi_2)$	9	57	34.569	2. Always west of 1.	$\Delta\lambda = \lambda_2 - \lambda_1$	7	20	16.446
$\Delta\phi_m = \frac{1}{2}(\phi_2-\phi_1)$		2	25.431		$\Delta\lambda_m = \frac{1}{2}\Delta\lambda$	3	40	08.223

$\sin\phi_m$ + 0.17295377 $\quad$ $\sin\Delta\phi_m$ + 0.00070507 $\quad$ $\sin\Delta\lambda$ + 0.12772073

$\cos\phi_m$ + 0.98492994 $\quad$ $\cos\Delta\phi_m$ + 0.99999975 $\quad$ $\sin\Delta\lambda_m$ + 0.06399152

$k = \sin\phi_m \cos\Delta\phi_m$ + 0.17295373 $\quad$ $K = \sin\Delta\phi_m \cos\phi_m$ + 0.00069444

$H = \cos^2\Delta\phi_m - \sin^2\phi_m = \cos^2\phi_m - \sin^2\Delta\phi_m$ + 0.97008649 $\quad$ $1 - L$ 0.99602708

$L = \sin^2\Delta\phi_m + H\sin^2\Delta\lambda_m$ + 0.00397292 $\quad$ $\cos d = 1 - 2L$ 0.99205416

d + 0.1261458534 $\quad$ $\sin d$ + 0.12581156 $\quad$ $T = d/\sin d$ + 1.00265710

$U = 2k^2/(1-L)$ + 0.060064618 $\quad$ $V = 2K^2/L$ + 0.000242767 $\quad$ $E = 60\cos d$ +59.52324960

$X = U + V$ + 0.060307385 $\quad$ $Y = U - V$ + 0.059821851 $\quad$ $D = 8(6 + T^2)$ +56.04257008

$A = 4T(16 + ET/15)$ + 80.12738460 $\quad$ $C = 2T - \frac{1}{2}(A + E)$ − 67.82000290 $\quad$ $B = -2D$ −112.08514016

$X(A + CX)$ + 4.58561299 $\quad$ $Y(B + EY)$ − 6.49212745 $\quad$ DXY + 0.20218475

$(TX - 3Y)$ − 0.118997925 $\quad$ $\delta f = -(f/4)(TX - 3Y)$ + 1.00853×10^{-4}

$T + \delta f$ + 1.00275795 $\quad$ $S_1 = a\sin d\,(T + \delta f)$ 804,665.223 meters

$\Sigma = X(A + CX) + Y(B + EY) + DXY$ − 1.70432971 $\quad$ $\delta f^2 = +(f^2/128)\,\Sigma$ − 1.53×10^{-7}

$T + \delta f + \delta f^2$ + 1.00275780 $\quad$ $S_2 = a\sin d\,(T + \delta f + \delta f^2)$ 804,665.102 meters

		°	'	"
$\sin(\alpha_2 + \alpha_1) = (K\sin\Delta\lambda)/L$ + 0.02232473	$\alpha_2 + \alpha_1$	361	16	45.188
$\sin(\alpha_2 - \alpha_1) = (k\sin\Delta\lambda)/(1-L)$ + 0.02217789	$\alpha_2 - \alpha_1$	178	43	45.107

$\frac{1}{2}(\delta\alpha_1 + \delta\alpha_2) = -(f/2)\,H\,(T + 1)\sin(\alpha_2 + \alpha_1)$ − 7.351613×10^{-5} $\quad$ $\delta\alpha_1$ − 7.350644×10^{-5}

$\frac{1}{2}(\delta\alpha_2 - \delta\alpha_1) = -(f/2)\,H\,(T - 1)\sin(\alpha_2 - \alpha_1)$ − $0.000969006 \times 10^{-5}$ $\quad$ $\delta\alpha_2$ − 7.352582×10^{-5}

	°	'	"		°	'	"
α_1	91	16	30.040	α_2	270	00	15.147
$\delta\alpha_1$			− 15.162	$\delta\alpha_2$			− 15.166
α_{1-2}	91	16	14.878	α_{2-1}	269	59	59.981

$\alpha_{1-2} = \alpha_1 + \delta\alpha_1$ $\quad$ $\alpha_{2-1} = \alpha_2 + \delta\alpha_2$

$d = 7°\ 13'\ 39''.450$

Line No. 6, see Tables 1 and 2. (Pages 65,66)

DISTANCE COMPUTING FORM — ANDOYER-LAMBERT
TYPE APPROXIMATION WITH SECOND ORDER TERMS
(No conversion to parametric latitudes)
Clarke Spheroid 1866, a = 6,378,206.4 meters
$f/2 = 0.00169503765$, $f/4 = 0.000847518825$, $f^2/128 = 0.0897860195 \times 10^{-6}$

	°	′	″			°	′	″
ϕ_1	44	54	28.507	1. Origin	λ_1	10	47	43.883
ϕ_2	40	00	00.000	2. Terminus	λ_2	18	00	00.000

$\sin\phi_1$ +.705 96946 2. west of 1. $\Delta\lambda = \lambda_2 - \lambda_1$ 7 12 16.117
$\cos\phi_1$ +.708 24228 $\sin\phi_2$ +.642 78761 $\sin\Delta\lambda$ +0.125 41095
$\tan\phi_1$ +.996 79091 $\cos\phi_2$ +.766 04444 $\cos\Delta\lambda$ +0.992 10491
$\tan\phi_2$ +.839 09963 $\cos d = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos\Delta\lambda$ +.992 05004
$M = \cos\phi_1 \tan\phi_2 - \sin\phi_1 \cos\Delta\lambda$ −.106 10993 $\cot u = M/\sin\Delta\lambda$ −.846 09916
$N = \cos\phi_2 \tan\phi_1 - \sin\phi_2 \cos\Delta\lambda$ +.125 89339 $\cot v = N/\sin\Delta\lambda$ +1.00368900
$\sin d = \cos\phi_1 \sin\Delta\lambda/\sin v = \cos\phi_2 \sin\Delta\lambda/\sin u$ +.125 84404 u 130° 14′ 04.316″
$\csc d$ +7.946 343 744 $\cot d$ +7.883 170629 v 44 53 40.246
$1 + \cos d$ +1.99205004 $1 - \cos d$ +.00794996 $\sin u$ +.763 40687
$(\sin\phi_1 + \sin\phi_2)^2$ +1.81914563 $(\sin\phi_1 - \sin\phi_2)^2$ $+3.991946 \times 10^{-3}$ $\sin v$ +.705 80373
$K_1 = (\sin\phi_1 + \sin\phi_2)^2/(1 + \cos d)$ +.913202777 $K_2 = (\sin\phi_1 - \sin\phi_2)^2/(1 - \cos d)$ +.502134098
$X = K_1 + K_2$ +1.415336875 $Y = K_1 - K_2$ +.411068679 XY +.581800660
X^2 +2.003178470 Y^2 +.168977459 d_r +.126178588 d_r^2 +.015921036
$A = 64d_r + 16d_r^2 \cot d$ +10.083561536 $D = 48 \sin d + 8d_r^2 \csc d$ +7.052626119
$B = -2D$ −14.105252238 $E = 30 \sin 2d$ +7.49061510 $\sin 2d$ +.24968717
$C = -(30d_r + 8d_r^2 \cot d + E/2)$ −8.534731134 AX +14.271636465
BY −5.798227405 CX^2 −17.096589655 DXY +4.103222531
EY^2 +1.265745106 $\Sigma = AX + BY + CX^2 + DXY + EY^2$ −3.254212958
$\delta d_f = -(f/4)(Xd_r - 3Y \sin d)$ -1.98265×10^{-5} $\delta d_f^2 = +(f^2/128)\Sigma$ -2.9218×10^{-7}
$d_r + \delta d_f$ +.126158762 $d_r + \delta d_f + \delta d_f^2$ +.126158469
$S(\delta d_f) = a(d_r + \delta d_f)$ 804,666.623 m $S(\delta d_{f^2}) = a(d_r + \delta d_f + \delta d_{f^2})$ 804,664.754 m

$T = d/\sin d$ 1.002658433

2u 260 28 08.632 2v 89 47 20.492
sin 2u −.986 19633 sin 2v +.999 99322
$U = (f/2)\cos^2\phi_1 \sin 2u$ -8.385065×10^{-4} $V = (f/2)\cos^2\phi_2 \sin 2v$ $+9.946822 \times 10^{-4}$
VT $+9.973265 \times 10^{-4}$ UT -8.407356×10^{-4}
$\delta u = VT - U$ $+18.35833 \times 10^{-4}$ $\delta v = -UT + V$ $-18.354178 \times 10^{-4}$
$+\delta u$ + 6 18.668 $+\delta v$ + 6 18.582
− u −130 14 04.316 + v +44 53 40.246

	+180 °	′	″		+180 °	′	″
α_{1-2}	49	52	14.352	α_{2-1}	224	59	58.828

$\alpha_{1-2} = \alpha_{uv} = 180° - u + \delta u$ $\alpha_{2-1} = \alpha_{vu} = 180° + v + \delta v$

Line No. 7, See Tables 1 and 2. True distance 804,664.771 meters.

DISTANCE COMPUTING FORM — ANDOYER-LAMBERT TYPE APPROXIMATION WITH SECOND ORDER TERMS

(No conversion to parametric latitudes)

Clarke Spheroid 1866, a = 6,378,206.4 meters

$f/2 = 0.00169503765$, $f/4 = 0.000847518825$, $f^2/128 = 0.0897860195 \times 10^{-6}$

ϕ_1 + 76° 00′ 26.803″ N 1. Origin λ_1 28° 42′ 03.567″ E

ϕ_2 + 70° 00′ 00.000″ N 2. Terminus λ_2 18° 00′ 00.000″ W

$\sin\phi_1$ +.970 32692 2. west of 1. $\Delta\lambda = \lambda_2 - \lambda_1$ 46 42 03.567

$\cos\phi_1$ +.241 79675 $\sin\phi_2$ +.93969262 $\sin\Delta\lambda$ +.727 78462

$\tan\phi_1$ +4.012 9858 $\cos\phi_2$ +.34202014 $\cos\Delta\lambda$ +.685 805 77

$\tan\phi_2$ +2.74747742 $\cos d = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos\Delta\lambda$ +.96852475

$M = \cos\phi_1 \tan\phi_2 - \sin\phi_1 \cos\Delta\lambda$ −.00112469 $\cot u = M/\sin\Delta\lambda$ −.00154536

$N = \cos\phi_2 \tan\phi_1 - \sin\phi_2 \cos\Delta\lambda$ +.728 07535 $\cot v = N/\sin\Delta\lambda$ +1.00039947

$\sin d = \cos\phi_1 \sin\Delta\lambda/\sin v = \cos\phi_2 \sin\Delta\lambda/\sin u$ +.248 91730 u 90° 05′ 18.753″

$\csc d$ +4.01739855 $\cot d$ +3.890 94993 v 44 59 18.810

$1 + \cos d$ +1.96852475 $1 - \cos d$ +.031 47525 $\sin u$ +.999 99880

$(\sin\phi_1 + \sin\phi_2)^2$ 3.648 17464 $(\sin\phi_1 - \sin\phi_2)^2$ 9.3846034×10^{-4} $\sin v$ +.70696556

$K_1 = (\sin\phi_1 + \sin\phi_2)^2/(1 + \cos d)$ +1.85325312 $K_2 = (\sin\phi_1 - \sin\phi_2)^2/(1 - \cos d)$ +.02981582

$X = K_1 + K_2$ +1.883 06894 $Y = K_1 - K_2$ +1.82343730 XY +3.433 65814

X^2 +3.545 94863 Y^2 +3.32492359 d_r .2515 622076 d_r^2 .0632835443

$A = 64d_r + 16d_r^2 \cot d$ +20.0397109 3 $D = 48 \sin d + 8d_r^2 \csc d$ +13.98191215

$B = -2D$ −27.96382430 $E = 30 \sin 2d$ +14.46495396 $\sin 2d$ +.482165132

$C = -(30d_r + 8d_r^2 \cot d + E/2)$ −16.74920803 AX +37.736 15722

BY −50.99028028 CX^2 −59.39183127 DXY +48.00910647

EY^2 +48.09486665 $\Sigma = AX + BY + CX^2 + DXY + EY^2$ +23.45801899

$\delta d_f = -(f/4)(Xd_r - 3Y \sin d)$ +.00075 25512 $\delta d_f^2 = +(f^2/128)\Sigma$ $+2.1062021 \times 10^{-6}$

$d_r + \delta d_f$ +.252314 7588 $d_r + \delta d_f + \delta d_f^2$ +.25231685

$S(\delta d_f) = a(d_r + \delta d_f)$ 1,609,315.609 m $S(\delta d_{f^2}) = a(d_r + \delta d_f + \delta d_{f^2})$ 1,609,329.043 m

$T = d/\sin d$ 1.010625647

2u 180° 10′ 37.506″ 2v 89° 58′ 37.620″

sin 2u −.003 09071 sin 2v +.999 99992

$U = (f/2)\cos^2\phi_1 \sin 2u$ $-3.0629403 \times 10^{-7}$ $V = (f/2)\cos^2\phi_2 \sin 2v$ $+1.9838 1725 \times 10^{-4}$

VT $+2.0038860 \times 10^{-4}$ UT $-3.095\ 4860 \times 10^{-7}$

$\delta u = VT - U$ $+2.006\ 9489 \times 10^{-4}$ $\delta v = -UT + V$ $+1.988\ 92322 \times 10^{-4}$

$+\delta u$ + 41.396 $+\delta v$ + 41.024

$-u$ − 90 05 18.753 $+v$ + 44 59 18.810

+180 ° ′ ″ +180 ° ′ ″

α_{1-2} 89 55 22.643 α_{2-1} 224 59 59.834

$\alpha_{1-2} = \alpha_{uv} = 180° - u + \delta u$ $\alpha_{2-1} = \alpha_{vu} = 180° + v + \delta v$

Line No. 8, See Tables 1 and 2. True distance 1,609,329.060 meters.

DISTANCE COMPUTING FORM — ANDOYER-LAMBERT TYPE APPROXIMATION WITH SECOND ORDER TERMS

(No conversion to parametric latitudes)

Clarke Spheroid 1866, a = 6,378,206.4 meters

$f/2 = 0.00169503765$, $f/4 = 0.000847518825$, $f^2/128 = 0.0897860195 \times 10^{-6}$

ϕ_1 27° 49′ 42.″130 1. Origin λ_1 32° 54′ 12.″991 E

ϕ_2 40° 00′ 00.″000 2. Terminus λ_2 18° 00′ 00.″000 W

$\sin\phi_1$ +.466 82458 2. west of 1. $\Delta\lambda = \lambda_2 - \lambda_1$ 50 54 12.997

$\cos\phi_1$ +.884 34994 $\sin\phi_2$ +.642 78761 $\sin\Delta\lambda$ +.776 08614

$\tan\phi_1$ +.527 87314 $\cos\phi_2$ +.766 04444 $\cos\Delta\lambda$ +.630 62691

$\tan\phi_2$ +.839 09963 $\cos d = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos\Delta\lambda$ +.727 28811

$M = \cos\phi_1 \tan\phi_2 - \sin\phi_1 \cos\Delta\lambda$ +.447 66557 $\cot u = M/\sin\Delta\lambda$ +.576 82459

$N = \cos\phi_2 \tan\phi_1 - \sin\phi_2 \cos\Delta\lambda$ −.000 984 88 $\cot v = N/\sin\Delta\lambda$ −.001 26903

$\sin d = \cos\phi_1 \sin\Delta\lambda/\sin v = \cos\phi_2 \sin\Delta\lambda/\sin u$ +.686 33229 u 60° 01′ 21.″339

$\csc d$ +1.45702018 $\cot d$ +1.059 67346 v 90 04 21.758

$1 + \cos d$ +1.72728811 $1 - \cos d$ +.272 71189 $\sin u$ +.866 22251

$(\sin\phi_1 + \sin\phi_2)^2$ +1.23123921 $(\sin\phi_1 - \sin\phi_2)^2$ +.030 96299 $\sin v$ +.999 99919

$K_1 = (\sin\phi_1 + \sin\phi_2)^2/(1 + \cos d)$ +.712 816352 $K_2 = (\sin\phi_1 - \sin\phi_2)^2/(1 - \cos d)$ +.113 537360

$X = K_1 + K_2$ +.826 35371 $Y = K_1 - K_2$ +.599 27899 XY +.495 21642

X^2 +.682 86045 Y^2 +.359 13531 d_r +.75643397 8 d_r^2 +.572 192348

$A = 64d_r + 16d_r^2 \cot d$ +58.113 16731 $D = 48 \sin d + 8d_r^2 \csc d$ +37.521 4887

$B = -2D$ −75.042 9774 $E = 30 \sin 2d$ +29.949 6783 $\sin 2d$ +.998 32261

$C = -(30d_r + 8d_r^2 \cot d + E/2)$ −42.51855 485 AX +48.0220 3141

BY −44.971 67970 CX^2 −29.034 23950 DXY +18.58125731

EY^2 +10.755 98700 $\Sigma = AX + BY + CX^2 + DXY + EY^2$ +3.353 35652

$\delta d_f = -(f/4)(Xd_r - 3Y \sin d)$ +.000515 9962 $\delta d_f^2 = +(f^2/128)\Sigma$ +.0000003011

$d_r + \delta d_f$ +.756 949974 $d_r + \delta d_f + \delta d_f^2$ +.756 950275

$S(\delta d_f) = a(d_r + \delta d_f)$ 4,827,983.169 m $S(\delta d_{f^2}) = a(d_r + \delta d_f + \delta d_{f^2})$ 4,827,985.088 m

$T = d/\sin d$ 1.102 139574

2u 120° 02′ 42.″678 2v 180° 08′ 43.″516

$\sin 2u$ +.865 63079 $\sin 2v$ −.002 53807

$U = (f/2) \cos^2\phi_1 \sin 2u$ .00114 752022 $V = (f/2) \cos^2\phi_2 \sin 2v$ -2.52459×10^{-6}

VT −.00000 278245 UT +.00126 472 745

$\delta u = VT - U$ −.001150302 67 $\delta v = -UT + V$ −.00126 725204

$+ \delta u$ − 3 57.267 $+ \delta v$ − 4 20.869

$- u$ −60 01 21.339 $+ v$ +90 04 21.758

+180 +180

α_{1-2} 119° 54′ 41.″394 α_{2-1} 270° 00′ 00.″889

$\alpha_{1-2} = \alpha_{uv} = 180° - u + \delta u$ $\alpha_{2-1} = \alpha_{vu} = 180° + v + \delta v$

Line No. 9, See Tables 1 and 2. True distance 4,827,984.247 meters.

DISTANCE COMPUTING FORM — ANDOYER-LAMBERT TYPE APPROXIMATION WITH SECOND ORDER TERMS

(No conversion to parametric latitudes)

Clarke Spheroid 1866, a = 6,378,206.4 meters

$f/2 = 0.00169503765$, $f/4 = 0.000847518825$, $f^2/128 = 0.0897860195 \times 10^{-6}$

	°	′	″				°	′	″
ϕ_1	35	18	45.644N	1.	Origin	λ_1	102	02	29.370E
ϕ_2	40	00	00.000N	2.	Terminus	λ_2	18	00	00.000N

$\sin \phi_1$ +.578 03821   2. west of 1. $\Delta\lambda = \lambda_2 - \lambda_1$ 120 02 29.370

$\cos \phi_1$ +.816 00970 $\sin \phi_2$ +.642 78761 $\sin \Delta\lambda$ +.865 66309

$\tan \phi_1$ +.708 37174 $\cos \phi_2$ +.766 04444 $\cos \Delta\lambda$ −.50062701

$\tan \phi_2$ +.839 09964 $\cos d = \sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2 \cos \Delta\lambda$ +.05861401

$M = \cos \phi_1 \tan \phi_2 - \sin \phi_1 \cos \Delta\lambda$ +.974 094 9862 $\cot u = M/\sin \Delta\lambda$ +1.12525877

$N = \cos \phi_2 \tan \phi_1 - \sin \phi_2 \cos \Delta\lambda$ +.864 4410722 $\cot v = N/\sin \Delta\lambda$ +.99858834 4

$\sin d = \cos \phi_1 \sin \Delta\lambda/\sin v = \cos \phi_2 \sin \Delta\lambda/\sin u$ +.99828072   u 41° 37′ 37.″186

$\csc d$ +1.00172224 $\cot d$ +.058 71496   v 45 02 25.691

$1 + \cos d$ +1.05861401 $1 - \cos d$ +.941 38599 $\sin u$ +.664 27848

$(\sin \phi_1 + \sin \phi_2)^2$ +1.49041568 $(\sin \phi_1 - \sin \phi_2)^2$ +.004 1924848 $\sin v$ +.707 60605

$K_1 = (\sin \phi_1 + \sin \phi_2)^2/(1 + \cos d)$ 1.407 893402 $K_2 = (\sin \phi_1 - \sin \phi_2)^2/(1 - \cos d)$ +.00445352 4

$X = K_1 + K_2$ +1.412346926 $Y = K_1 - K_2$ +1.403 439878   XY +1.982 143998

X^2 +1.99472383 9 Y^2 +1.969643491 d_r +1.512 148751 d_r^2 +2.28659384 5

$A = 64d_r + 16d_r^2 \cot d$ +98.925 636 322 $D = 48 \sin d + 8d_r^2 \csc d$ +66.2417 2983

$B = -2D$ −132.48345966 $E = 30 \sin 2d$ +3.510794166 $\sin 2d$ +.117026472

$C = -(30d_r + 8d_r^2 \cot d + E/2)$ −48.19381774   AX +139.717318359

BY −185.932 57052 CX^2 −96.133357138   DXY +131.300 64720

EY^2 +6.915 012877 $\Sigma = AX + BY + CX^2 + DXY + EY^2$ −4.132 94922

$\delta d_f = -(f/4)(Xd_r - 3Y \sin d)$ +.0017522162 $\delta d_f^2 = +(f^2/128)\Sigma$ −.0000003711

$d_r + \delta d_f$ +1.513 900913 $d_r + \delta d_f + \delta d_f^2$ +1.513900542

$S(\delta d_f) = a(d_r + \delta d_f)$ 9,655,972.492 m $S(\delta d_{f^2}) = a(d_r + \delta d_f + \delta d_{f^2})$ 9,655,970.126 m

$T = d/\sin d$ +1.51475305

	°	′	″		°	′	″
2u	83	15	14.382	2v	90	04	51.416

$\sin 2u$ +.993 07665 $\sin 2v$ +.999 99900

$U = (f/2) \cos^2\phi_1 \sin 2u$ +.00112 0864 $V = (f/2) \cos^2\phi_2 \sin 2v$ +.00099 46879

VT 5′ 10.″780   UT 5′ 50.″203

$\delta u = VT - U$ 1′ 19.″585 $\delta v = -UT + V$ 2′ 25.″034

$+ \delta u$ + 1′ 19.″585 $+ \delta v$ − 2′ 25.034

$- u$ −41 37 37.191 $+ v$ +45 02 25.708

+180   +180

α_{1-2} 138° 23′ 42.″394 α_{2-1} 225° 00′ 00.″674

$\alpha_{1-2} = \alpha_{uv} = 180° - u + \delta u$ $\alpha_{2-1} = \alpha_{vu} = 180° + v + \delta v$

Line No.10, See Tables 1 and 2. True distance 9,655,969.751 meters.

DISTANCE COMPUTING FORM — ANDOYER-LAMBERT TYPE APPROXIMATION WITH SECOND ORDER TERMS

(No conversion to parametric latitudes)

Clarke Spheroid 1866, a = 6,378,206.4 meters

$f/2 = 0.00169503765$, $f/4 = 0.000847518825$, $f^2/128 = 0.0897860195 \times 10^{-6}$

ϕ_1 2° 55′ 17.425″(N) 1. Origin λ_1 70° 50′ 04.869″E

ϕ_2 70 00 00.000 (N) 2. Terminus λ_2 18 00 00.000W

$\sin \phi_1$ +.050 96783 2. west of 1. $\Delta\lambda = \lambda_2 - \lambda_1$ +88 50 04.869

$\cos \phi_1$ .998 70029 $\sin \phi_2$ +.93969262 $\sin \Delta\lambda$ +.999 79318

$\tan \phi_1$ .051 03416 $\cos \phi_2$ +.34202014 $\cos \Delta\lambda$ +.020 33717

$\tan \phi_2$ 2.747 47742 $\cos d = \sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2 \cos \Delta\lambda$ +.05484078

$M = \cos \phi_1 \tan \phi_2 - \sin \phi_1 \cos \Delta\lambda$ +2.742869955 $\cot u = M/\sin \Delta\lambda$ +2.743437352

$N = \cos \phi_2 \tan \phi_1 - \sin \phi_2 \cos \Delta\lambda$ −.001 6559781 $\cot v = N/\sin \Delta\lambda$ −.001656321

$\sin d = \cos \phi_1 \sin \Delta\lambda / \sin v = \cos \phi_2 \sin \Delta\lambda / \sin u$ +.99849511 u 20° 01′ 37.607″

$\csc d$ +1.00150716 $\cot d$ +.05492343 v 90 05 41.640

$1 + \cos d$ +1.054 84078 $1 - \cos d$ +.945 15922 $\sin u$ +.342 46478

$(\sin \phi_1 + \sin \phi_2)^2$ +.981408127 $(\sin \phi_1 - \sin \phi_2)^2$ +.789831752 $\sin v$ +.99999863

$K_1 = (\sin \phi_1 + \sin \phi_2)^2/(1 + \cos d)$ +.930385083 $K_2 = (\sin \phi_1 - \sin \phi_2)^2/(1 - \cos d)$ +.835659998

$X = K_1 + K_2$ +1.766045081 $Y = K_1 - K_2$ +.094725085 XY +.16728877

X^2 +3.1189 1523 Y^2 +.008972 8417 d_r 1.515928018 d_r^2 +2.29803776

$A = 64d_r + 16d_r^2 \cot d$ +99.038851009 $D = 48 \sin d + 8d_r^2 \csc d$ +66.33977544

$B = -2D$ −132.67955088 $E = 30 \sin 2d$ +3.2854950 $\sin 2d$ +.10951650

$C = -(30d_r + 8d_r^2 \cot d + E/2)$ −48.13031697 AX +174.907075654

BY −12.568081737 CX^2 −150.114378622 DXY +11.097899435

EY^2 +.029480237 $\Sigma = AX + BY + CX^2 + DXY + EY^2$ +23.35199496

$\delta d_f = -(f/4)(Xd_r - 3Y \sin d)$ −.0020284936 $\delta d_f^2 = +(f^2/128)\Sigma$ +.0000020966 8

$d_r + \delta d_f$ +1.513899524 $d_r + \delta d_f + \delta d_f^2$ +1.513901621

$S(\delta d_f) = a(d_r + \delta d_f)$ 9,655,963.633 m $S(\delta d_{f^2}) = a(d_r + \delta d_f + \delta d_{f^2})$ 9,655,977.008 m

$T = d/\sin d$ +1.5182127 6

2u 40 03 15.214 2v 180° 11′ 23.280″

$\sin 2u$ +.643 51232 $\sin 2v$ −.00331263

$U = (f/2)\cos^2\phi_1 \sin 2u$ +1.087944 × 10⁻³ $V = (f/2)\cos^2\phi_2 \sin 2v$ −6.56834 × 10⁻⁷

VT −9.972138 × 10⁻⁷ UT +1.6517306 × 10⁻³

$\delta u = VT - U$ −.00108 8941 $\delta v = -UT + V$ −.00165 2 3874

$+\delta u$ − 03 44.610 $+\delta v$ − 05 40.829

$-u$ − 20 01 37.607 $+v$ + 90 05 41.640

+180° ′ ″ +180° ′ ″

α_{1-2} 159 54 37.783 α_{2-1} 270 00 00.811

$\alpha_{1-2} = \alpha_{uv} = 180° - u + \delta u$ $\alpha_{2-1} = \alpha_{vu} = 180° + v + \delta v$

Line No.11, See Tables 1 and 2. True distance 9,655,977.148 meters.

DISTANCE COMPUTING FORM, ANDOYER-LAMBERT TYPE APPROXIMATION
WITH SECOND ORDER TERMS

(No conversion to parametric latitudes)

Clarke Spheroid 1866, a = 6,378,206.4 meters

f/2 = 0.00169503765, f/4 = 0.000847518825, $f^2/128 = 0.0897860195 \times 10^{-6}$

1 radian = 206,264.8062 seconds

ϕ_1 70 00 00.0   1. Origin λ_1

ϕ_2 69 46 36.574   2. Terminus λ_2

$\phi_m = \frac{1}{2}(\phi_2+\phi_1)$ 69 53 18.287   2. Always west of 1. $\Delta\lambda = \lambda_2 - \lambda_1$ 15 39 28.298

$\Delta\phi_m = \frac{1}{2}(\phi_2-\phi_1)$ − 6 41.713 $\Delta\lambda_m = \frac{1}{2}\Delta\lambda$ 7 49 44.149

sin ϕ_m +.93902474   sin $\Delta\phi_m$ −.00194756   sin $\Delta\lambda$ +.26989234

cos ϕ_m +.34384960   cos $\Delta\phi_m$ +.99999810   sin $\Delta\lambda_m$ +.13621582

k = sin ϕ_m cos $\Delta\phi_m$ +.939022956   K = sin $\Delta\phi_m$ cos ϕ_m −.00066966727

$H = \cos^2\Delta\phi_m - \sin^2\phi_m = \cos^2\phi_m - \sin^2\Delta\phi_m$ +.118228745   1−L +.997802502

$L = \sin^2\Delta\phi_m + H\sin^2\Delta\lambda_m$ .002197498   cos d = 1−2L +.995605004

d +.093789593   sin d + .09365191   T = d/sin d + 1.001467661

$U = 2k^2/(1-L)$ +1.767412109 $V = 2K^2/L$ +.000408150 4

X = U + V +1.767820259   Y = U − V +1.767003959   XY +3.123745396

X^2 +3.125188468 Y^2 +3.122302991   E = 60 cos d +59.7363002 4

$A = 4[16T + (E/15)T^2]$ +80.07040344 $D = 8(6+T^2)$ +56.023499808

B = −2D −112.046999616   C = 2T − ½(A + E) −67.9004165 2

AX +141.550081348   BY −197.98749188 7 CX^2 −212.2015 98681

DXY +175.003149599 EY^2 +186.514828911 $\delta_f = -(f/4)(TX - 3Y)$ +.00299224747

T + δ_f +1.004459909 $S_1 = a \sin d\,(T + \delta_f)$ 599,995.255 m

$\delta_{f^2} = +(f^2/128)(AX + BY + CX^2 + DXY + EY^2)$ $+8.33923 \times 10^{-6}$

T + δ_f + δ_{f^2} +1.004468248 $S_2 = a \sin d\,(T + \delta_f + \delta_{f^2})$ 600,000.236 m

$\sin(\alpha_2+\alpha_1) = (K \sin\Delta\lambda)/L$ −.08224726 $\alpha_2 + \alpha_1$ 355° 16′ 56.099″

$\sin(\alpha_2-\alpha_1) = (k \sin\Delta\lambda)/(1-L)$ +.25399325 $\alpha_2 - \alpha_1$ 165 17 09.821

$\frac{1}{2}(\delta\alpha_1 + \delta\alpha_2) = -(f/2)H(T+1)\sin(\alpha_1+\alpha_2)$ $+.3298925 \times 10^{-4}$ $\delta\alpha_1$ $+.3306396 \times 10^{-4}$

$\frac{1}{2}(\delta\alpha_2 - \delta\alpha_1) = -(f/2)H(T-1)\sin(\alpha_2-\alpha_1)$ $-.0007471 \times 10^{-4}$ $\delta\alpha_2$ $+.3291454 \times 10^{-4}$

α_1 260° 17′ 02.960″ α_2 94° 59′ 53.139″

$\delta\alpha_1$ + 00 06.820 $\delta\alpha_2$ + 00 06.789

α_{1-2} 260 17 09.780 α_{2-1} 94 59 59.928

$\alpha_{1-2} = +\alpha_1 + \delta\alpha_1$ $\alpha_{2-1} = +\alpha_2 + \delta\alpha_2$

d = 5° 22′ 25.444″   True distance 600,000.00 meters

True Azimuths 260 17 09.79   95 00 00.000

Line No. 12

DISTANCE COMPUTING FORM, ANDOYER-LAMBERT TYPE APPROXIMATION WITH SECOND ORDER TERMS

(No conversion to parametric latitudes)

Clarke Spheroid 1866, a = 6,378,206.4 meters

$f/2 = 0.00169503765$, $f/4 = 0.000847518825$, $f^2/128 = 0.0897860195 \times 10^{-6}$

1 radian = 206,264.8062 seconds

ϕ_1 60° 00' 00.000" 1. Origin λ_1

ϕ_2 54° 18' 59.319" 2. Terminus λ_2

$\phi_m = \frac{1}{2}(\phi_2+\phi_1)$ 57°09'29."659 2. Always west of 1. $\Delta\lambda = \lambda_2 - \lambda_1$ +10. 37 10.172

$\Delta\phi_m = \frac{1}{2}(\phi_2 - \phi_1)$ −2° 50' 30."340 $\Delta\lambda_m = \frac{1}{2}\Delta\lambda$ +5 18 35.086

$\sin\phi_m$ +.840 17154 $\sin\Delta\phi_m$ −.049 57776 $\sin\Delta\lambda$ +.184 28574

$\cos\phi_m$ +.542 32073 $\cos\Delta\phi_m$ +.998 77027 $\sin\Delta\lambda_m$ +.092 53996

$k = \sin\phi_m \cos\Delta\phi_m$ +.839 138356 $K = \sin\Delta\phi_m \cos\phi_m$ −.02688 7047

$H = \cos^2\Delta\phi_m - \sin^2\phi_m = \cos^2\phi_m - \sin^2\Delta\phi_m$ +.29165383 $1-L$ +.995 044426

$L = \sin^2\Delta\phi_m + H\sin^2\Delta\lambda_m$ +.004 9555 7392 $\cos d = 1-2L$ +.99008885 2

d + .140 9083457 sin d + .140 44242 $T = d/\sin d$ + 1.0033 16844

$U = 2k^2/(1-L)$ +1.41532 008 $V = 2K^2/L$ +.291 757 648

$X = U+V$ +1.707 077728 $Y = U-V$ +1.123562432 XY +1.918 008404

X^2 +2.914 114 369 Y^2 +1.262 392539 $E = 60 \cos d$ +59.405 33112

$A = 4[16T + (E/15)T^2]$ +80.158 96096 $D = 8(6+T^2)$ +56.053157 512

$B = -2D$ −112.106315024 $C = 2T - \frac{1}{2}(A+E)$ −67.775 512 35

AX +136.8375 76454 BY −125.958443924 CX^2 −197.50559 4405

DXY +107.510 427175 EY^2 +74.99284678 3 $\delta_f = -(f/4)(TX - 3Y)$ +.001405142

$T + \delta_f$ +1.004721986 $S_1 = a \sin d (T+\delta_f)$ 900 000.559 m

$\delta_{f^2} = +(f^2/128)(AX + BY + CX^2 + DXY + EY^2)$ −.370205 X 10 −6

$T + \delta_f + \delta_{f^2}$ +1.004721616 $S_2 = a\sin d(T + \delta_f + \delta_{f^2})$ 900,000.228 m

$\sin(\alpha_2+\alpha_1) = (K\sin\Delta\lambda)/L$ −.99986388 $\alpha_2+\alpha_1$ 270° 56' 43.429"

$\sin(\alpha_2-\alpha_1) = (k\sin\Delta\lambda)/(1-L)$ +.155 41139 $\alpha_2-\alpha_1$ 171 03 33.636

$\frac{1}{2}(\delta\alpha_1+\delta\alpha_2) = -(f/2)H(T+1)\sin(\alpha_1+\alpha_2)$ +9.90233366 X 10 −4 $\delta\alpha_1$ +9.90488199 X 10 −4

$\frac{1}{2}(\delta\alpha_2-\delta\alpha_1) = -(f/2)H(T-1)\sin(\alpha_2-\alpha_1)$ −0.0025 4833 X 10 −4 $\delta\alpha_2$ +9.899 78533 X 10 −4

α_1 49° 56' 34."896 α_2 231° 00' 08."532

$\delta\alpha_1$ + 03 24.303 $\delta\alpha_2$ + 03 24.198

α_{1-2} 49 59 59.199 α_{2-1} 231 03 32.730

$\alpha_{1-2} = +\alpha_1 + \delta\alpha_1$ $\alpha_{2-1} = +\alpha_2 + \delta\alpha_2$

d = 8° 04' 24."412 True distance 900,000.00 meters

True Azimuths 50° 00' 00."000 231° 03' 33."54

Line No. 13

DISTANCE COMPUTING FORM, ANDOYER-LAMBERT TYPE APPROXIMATION
WITH SECOND ORDER TERMS

(No conversion to parametric latitudes)

Clarke Spheroid 1866, a = 6,378,206.4 meters

f/2 = 0.00169503765, f/4 = 0.000847518825, $f^2/128 = 0.0897860195 \times 10^{-6}$

1 radian = 206,264.8062 seconds

ϕ_1 19° 51′ 31.432″ 1. Origin λ_1 ____

ϕ_2 25 12 03.231 2. Terminus λ_2 ____

$\phi_m = \frac{1}{2}(\phi_2+\phi_1)$ 22 31 47.332 2. Always west of 1. $\Delta\lambda = \lambda_2 - \lambda_1$ 7 35 26.377

$\Delta\phi_m = \frac{1}{2}(\phi_2 - \phi_1)$ 2 40 15.899 $\Delta\lambda_m = \frac{1}{2}\Delta\lambda$ 3 47 43.188

sin ϕ_m +.383 16413 sin $\Delta\phi_m$ +.046 60231 sin $\Delta\lambda$ +.132 09481

cos ϕ_m .923 68037 cos $\Delta\phi_m$ +.998 91352 sin $\Delta\lambda_m$ +.066 19257

k = sin ϕ_m cos $\Delta\phi_m$ +.382 747 830 K = sin $\Delta\phi_m$ cos ϕ_m +.043 045 634

H = $\cos^2\Delta\phi_m - \sin^2\phi_m = \cos^2\phi_m - \sin^2\Delta\phi_m$ +.851 01347 1−L +.994 099547

$L = \sin^2\Delta\phi_m + H\sin^2\Delta\lambda_m$ +.005 900 453 cos d = 1−2L +.988 19909

d + .153 780 3447 sin d + .153 17496 T = d/sin d + 1.003 952243

$U = 2k^2/(1-L)$ +.294 730848 $V = 2K^2/L$ +.628 062 491

X = U + V +.922 793339 Y = U − V −.333 331643 XY −.307 596220

X^2 +.851 547 547 Y^2 +.111 109 984 E = 60 cos d +59.291945400

$A = 4[16T + (E/15)T^2]$ +80.189355264 $D = 8(6+T^2)$ +56.063360848

B = −2D −112.126 72170 $C = 2T - \frac{1}{2}(A+E)$ −67.732 745 846

AX +73.998202 893 BY +37.375 384256 CX^2 −57.677653580

DXY −17.244 877 878 EY^2 +6.587927105 $\delta_f = -(f/4)(TX - 3Y)$ −.0016326902

T + δ_f +1.002 319 553 $S_1 = a \sin d\,(T+\delta_f)$ 979,247.671 m

$\delta_{f^2} = +(f^2/128)(AX + BY + CX^2 + DXY + EY^2)$ $+3.8643 \times 10^{-6}$

T + δ_f + δ_{f^2} 1.002 323 417 $S_2 = a \sin d\,(T + \delta_f + \delta_{f^2})$ 979,251.446 m

$\sin(\alpha_2+\alpha_1) = (K \sin\Delta\lambda)/L$ +.963 67259 $\alpha_2 + \alpha_1$ 434° 30′ 32.531″

$\sin(\alpha_2-\alpha_1) = (k \sin\Delta\lambda)/(1-L)$ +.050 85909 $\alpha_2 - \alpha_1$ 177 05 05.131

$\frac{1}{2}(\delta\alpha_1+\delta\alpha_2) = -(f/2)H(T+1)\sin(\alpha_1+\alpha_2)$ $-2.78568918 \times 10^{-3}$ $\delta\alpha_1$ $-2.78539923 \times 10^{-3}$

$\frac{1}{2}(\delta\alpha_2-\delta\alpha_1) = -(f/2)H(T-1)\sin(\alpha_2-\alpha_1)$ $-.00028995 \times 10^{-3}$ $\delta\alpha_2$ $-2.78597913 \times 10^{-3}$

α_1 128° 42′ 43.700″ α_2 305° 47′ 48.831″

$\delta\alpha_1$ − 0 9 34.530 $\delta\alpha_2$ − 9 34.649

α_{1-2} 128 33 09.170 α_{2-1} 305 38 14.182

$\alpha_{1-2} = +\alpha_1 + \delta\alpha_1$ $\alpha_{2-1} = +\alpha_2 + \delta\alpha_2$

d = 8° 48′ 39.473″ True distance 979,251.25 meters

True Azimuths 128 33 08.34 305 38 13.25

Line No. 14

DISTANCE COMPUTING FORM, ANDOYER-LAMBERT TYPE APPROXIMATION
WITH SECOND ORDER TERMS

(No conversion to parametric latitudes)

Clarke Spheroid 1866, a = 6,378,206.4 meters

$f/2 = 0.00169503765$, $f/4 = 0.000847518825$, $f^2/128 = 0.0897860195 \times 10^{-6}$

1 radian = 206,264.8062 seconds

ϕ_1 59° 30' 12.0" — 1. Origin — λ_1

ϕ_2 50° 00' 03.8" — 2. Terminus — λ_2

$\phi_m = \frac{1}{2}(\phi_2+\phi_1)$ +54 45 07.9 — 2. Always west of 1. — $\Delta\lambda = \lambda_2 - \lambda_1$ 9 55 01.000

$\Delta\phi_m = \frac{1}{2}(\phi_2 - \phi_1)$ −4 45 04.1 — $\Delta\lambda_m = \frac{1}{2}\Delta\lambda$ 4 57 30.500

$\sin\phi_m$ +0.816 66 366 — $\sin\Delta\phi_m$ −.08282801 — $\sin\Delta\lambda$ +.172 22043

$\cos\phi_m$ +0.577 11392 — $\cos\Delta\phi_m$ +.99656386 — $\sin\Delta\lambda_m$ +.086 43369

$k = \sin\phi_m \cos\Delta\phi_m$ +.813 857 489 — $K = \sin\Delta\phi_m \cos\phi_m$ −.047 801198

$H = \cos^2\Delta\phi_m - \sin^2\phi_m = \cos^2\phi_m - \sin^2\Delta\phi_m$ +.326199995 5 — $1-L$ +.990 702 5515

$L = \sin^2\Delta\phi_m + H\sin^2\Delta\lambda_m$ +.009297 4485 — $\cos d = 1-2L$ +.981 405 103

d + .193 146 6435 — $\sin d$ + .191 94797 — $T = d/\sin d$ + 1.006244783

$U = 2k^2/(1-L)$ +1.3371602024 — $V = 2K^2/L$ +.491 522 922 65

$X = U+V$ +1.82868312 5 — $Y = U-V$ +.845 6372798 — XY +1.546 402623

X^2 +3.344081972 — Y^2 +.715102 4090 — $E = 60\cos d$ +58.88430618

$A = 4[16T + (E/15)T^2]$ +80.298877292 — $D = 8(6+T^2)$ +56.100228 504

$B = -2D$ −112.200457008 — $C = 2T - \frac{1}{2}(A+E)$ −67.579102 170

AX +146.84120185 7 — BY −94.88088933 4 — CX^2 −225.99005 7251

DXY +86.753540503 — EY^2 +42.108309202 — $\delta_f = -(f/4)(TX-3Y)$ +.000590559

$T+\delta_f$ +1.006 835 342 — $S_1 = a\sin d\,(T+\delta_f)$ 1,232,652.169 m

$\delta_{f^2} = +(f^2/128)(AX + BY + CX^2 + DXY + EY^2)$ −4.055 4455 × 10⁻⁶

$T+\delta_f+\delta_{f^2}$ 1.006 831287 — $S_2 = a\sin d\,(T+\delta_f+\delta_{f^2})$ 1,232,647.205 m

$\sin(\alpha_2+\alpha_1) = (K\sin\Delta\lambda)/L$ −.885 44 108 — $\alpha_2+\alpha_1$ 242° 18' 21.056"

$\sin(\alpha_2-\alpha_1) = (k\sin\Delta\lambda)/(1-L)$ +.141 47828 — $\alpha_2-\alpha_1$ 171 51 59.771

$\frac{1}{2}(\delta\alpha_1+\delta\alpha_2) = -(f/2)H(T+1)\sin(\alpha_1+\alpha_2)$ +9.822157 × 10⁻⁴ — $\delta\alpha_1$ +9.827042 × 10⁻⁴

$\frac{1}{2}(\delta\alpha_2-\delta\alpha_1) = -(f/2)H(T-1)\sin(\alpha_2-\alpha_1)$ −.004885 × 10⁻⁴ — $\delta\alpha_2$ +9.817272 × 10⁻⁴

α_1 35° 13' 10.643" — α_2 207° 05' 10.414"

$\delta\alpha_1$ + 3 22.697 — $\delta\alpha_2$ + 3 22.496

α_{1-2} 35 16 33.340 — α_{2-1} 207 08 32.910

$\alpha_{1-2} = +\alpha_1 + \delta\alpha_1$ — $\alpha_{2-1} = +\alpha_2 + \delta\alpha_2$

d = 11° 03' 59.355" — True distance 1,232,647.21 meters

True Azimuths 35 16 34.25 — 207 08 33.82

Line No. 15

DISTANCE COMPUTING FORM, ANDOYER-LAMBERT TYPE APPROXIMATION WITH SECOND ORDER TERMS

(No conversion to parametric latitudes)

Clarke Spheroid 1866, a = 6,378,206.4 meters

$f/2 = 0.00169503765$, $f/4 = 0.000847518825$, $f^2/128 = 0.0897860195 \times 10^{-6}$

1 radian = 206,264.8062 seconds

ϕ_1 8° 58′ 25.0″	1. PANAMA	λ_1 79° 34′ 24.0″
ϕ_2 21 26 06.0	2. HAWAII	λ_2 158 01 33.0
$\phi_m = \frac{1}{2}(\phi_2+\phi_1)$ 15 12 15.5	2. Always west of 1.	$\Delta\lambda = \lambda_2 - \lambda_1$ 78 27 09.0
$\Delta\phi_m = \frac{1}{2}(\phi_2-\phi_1)$ 6 13 50.5		$\Delta\lambda_m = \frac{1}{2}\Delta\lambda$ 39 13 34.5
$\sin\phi_m$ +.26226170	$\sin\Delta\phi_m$ +.10853193	$\sin\Delta\lambda$ +.97975909
$\cos\phi_m$ +.96499679	$\cos\Delta\phi_m$ +.99409297	$\sin\Delta\lambda_m$ +.63238428
$k = \sin\phi_m \cos\Delta\phi_m$ +.260712512		$K = \sin\Delta\phi_m \cos\phi_m$ +.104732963
$H = \cos^2\Delta\phi_m - \sin^2\phi_m = \cos^2\phi_m - \sin^2\Delta\phi_m$ +.919439630		$1-L$ +.620527830
$L = \sin^2\Delta\phi_m + H\sin^2\Delta\lambda_m$ +.379472170		$\cos d = 1-2L$ +.241055660
d + 1.327342885	sin d + +.97051129	$T = d/\sin d$ + 1.367673822
$U = 2k^2/(1-L)$ +.2190748283		$V = 2K^2/L$ +.0578118469
$X = U+V$ +.2768866752	$Y = U-V$ .1612629814	XY +.0446515771
X^2 +.0766662309	Y^2 +.0260057492	$E = 60\cos d$ +14.4633396
$A = 4[16T+(E/15)T^2]$ +94.74556060		$D = 8(6+T^2)$ +62.964253464
$B = -2D$ −125.928506928		$C = 2T - \frac{1}{2}(A+E)$ −51.869102456
AX +26.233783264	BY −20.307606466	CX^2 −3.976608586
DXY +2.811452821	EY^2 +0.376129982	$\delta_f = -(f/4)(TX-3Y)$ $+8.90728 \times 10^{-5}$
$T+\delta_f$ +1.367762895		$S_1 = a\sin d\,(T+\delta_f)$ 8,466,618.258 m
$\delta_{f^2} = +(f^2/128)(AX+BY+CX^2+DXY+EY^2)$ $+4.6124 \times 10^{-7}$		
$T+\delta_f+\delta_{f^2}$ +1.367763356		$S_2 = a\sin d\,(T+\delta_f+\delta_{f^2})$ 8,466,621.112 m
$\sin(\alpha_2+\alpha_1) = (K\sin\Delta\lambda)/L$ +.27041001		$\alpha_2+\alpha_1$ 375° 41′ 19.197″
$\sin(\alpha_2-\alpha_1) = (k\sin\Delta\lambda)/(1-L)$ +.41164222		$\alpha_2-\alpha_1$ 155 41 31.161
$\frac{1}{2}(\delta\alpha_1+\delta\alpha_2) = -(f/2)H(T+1)\sin(\alpha_1+\alpha_2)$ −.000997808513		$\delta\alpha_1$ $-.761931734 \times 10^{-3}$
$\frac{1}{2}(\delta\alpha_2-\delta\alpha_1) = -(f/2)H(T-1)\sin(\alpha_2-\alpha_1)$ −.000235876779		$\delta\alpha_2$ $-1.233685292 \times 10^{-3}$
α_1 109° 59′ 54.018″		α_2 265° 41′ 25.179″
$\delta\alpha_1$ − 2 37.160		$\delta\alpha_2$ − 4 14.466
α_{1-2} 109 57 16.858		α_{2-1} 265 37 10.713
$\alpha_{1-2} = +\alpha_1 + \delta\alpha_1$		$\alpha_{2-1} = +\alpha_2 + \delta\alpha_2$
d = ° ′ ″		True distance 8,466,621.01 meters
True Azimuths 109 57 17.41		265 37 10.59

Line No. 16

DISTANCE COMPUTING FORM, ANDOYER-LAMBERT TYPE APPROXIMATION
WITH SECOND ORDER TERMS

(No conversion to parametric latitudes)

Clarke Spheroid 1866, a = 6,378,206.4 meters

$f/2 = 0.00169503765$, $f/4 = 0.000847518825$, $f^2/128 = 0.0897860195 \times 10^{-6}$

1 radian = 206,264.8062 seconds

ϕ_1 55 45 19.5 (N) 1. MOSCOW λ_1 −37 34 15.450 (E)

ϕ_2 −33 56 03.5 (S) 2. CAPE OF GOOD HOPE λ_2 −18 28 41.400 (E)

$\phi_m = \frac{1}{2}(\phi_2+\phi_1)$ +10 54 38.0 2. Always west of 1. $\Delta\lambda = \lambda_2 - \lambda_1$ +19 05 34.050

$\Delta\phi_m = \frac{1}{2}(\phi_2-\phi_1)$ −44 50 41.5 $\Delta\lambda_m = \frac{1}{2}\Delta\lambda$ +9 32 47.025

$\sin\phi_m$ +.189 27635 $\sin\Delta\phi_m$ −.705 18957 $\sin\Delta\lambda$ +.327 09901

$\cos\phi_m$ +.98192386 $\cos\Delta\phi_m$ +.709 01881 $\sin\Delta\lambda_m$ +.165 84631

$k = \sin\phi_m \cos\Delta\phi_m$ +.134 20049 $K = \sin\Delta\phi_m \cos\phi_m$ −.692 44246

$H = \cos^2\Delta\phi_m - \sin^2\phi_m = \cos^2\phi_m - \sin^2\Delta\phi_m$ +.466 88214 $1-L$ +.489 86609

$L = \sin^2\Delta\phi_m + H\sin^2\Delta\lambda_m$ +.510 13391 $\cos d = 1-2L$ −.02026782

d + 1.591065538 $\sin d$ + .99979459 $T = d/\sin d$ + 1.591392 42

$U = 2k^2/(1-L)$ +.073529368 $V = 2K^2/L$ +1.879806657

$X = U+V$ +1.953336025 $Y = U-V$ −1.806277289 XY −3.528266500

X^2 +3.81552 1627 Y^2 +3.262637645 $E = 60\cos d$ −1.216069200

$A = 4[16T + (E/15)T^2]$ +101.027853152 $D = 8(6+T^2)$ +68.260238672

$B = -2D$ −136.52047734 $C = 2T - \frac{1}{2}(A+E)$ −46.723 10712

AX +197.341345184 BY +246.5938 3763 CX^2 −178.273025697

DXY −240.840313381 EY^2 −3.967593151 $\delta_f = -(f/4)(TX-3Y)$ −.00722 7095

$T+\delta_f$ +1.584165325 $S_1 = a\sin d\,(T+\delta_f)$ 10,102,057.925 m

$\delta_{f^2} = +(f^2/128)(AX + BY + CX^2 + DXY + EY^2)$ $+1.87242 \times 10^{-6}$

$T+\delta_f+\delta_{f^2}$ +1.584167197 $S_2 = a\sin d\,(T+\delta_f+\delta_{f^2})$ 10,102,069.863 m

$\sin(a_2+a_1) = (K\sin\Delta\lambda)/L$ −.443 99566 a_2+a_1 206 21 32.759

$\sin(a_2-a_1) = (k\sin\Delta\lambda)/(1-L)$ +.089 60989 a_2-a_1 174 51 31.807

$\frac{1}{2}(\delta a_1+\delta a_2) = -(f/2)H(T+1)\sin(a_1+a_2)$ $+9.1053893 \times 10^{-4}$ δa_1 $+9.534779 \times 10^{-4}$

$\frac{1}{2}(\delta a_2-\delta a_1) = -(f/2)H(T-1)\sin(a_2-a_1)$ $-.4193902 \times 10^{-4}$ δa_2 $+8.685999 \times 10^{-4}$

	°	′	″		°	′	″
a_1	15	45	00.476	a_2	190	36	32.283
δa_1	+	3	16.463	δa_2	+	2	59.162
a_{1-2}	15	48	16.939	a_{2-1}	190	39	31.445

$a_{1-2} = +a_1 + \delta a_1$ $a_{2-1} = +a_2 + \delta a_2$

d = ° ′ ″ True distance 10,102,069.06 meters

True Azimuths 15 48 17.674 190 39 32.208

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13. ABSTRACT

The principal objective of this study was an evaluation of the formulas basic to the geodetic inverse solution for distance computations used by the U. S. Naval Oceanographic Office in loran-type charting. The adequacy of the formulas for past requirements was verified but, in anticipation of future requirements, they were modified to give geodesic distances and azimuths between any two points on the reference ellipsoid to uncertainties of less than a meter and a second respectively.

During the study, associated geometrical configurations were developed or studied: latitudes associated with the auxiliary sphere-spheroid configuration; a spherical rectangular coordinate system on the auxiliary sphere with hyperbolic loci referenced to it; and geometrical quantities associated with arc distance, such as chord length, dip of the chord, maximum separation of chord and arc, and the geographical position of the point of maximum separation. The formulas with their derivations are presented. (U)

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	ROLE	WT	ROLE	WT	ROLE	WT
Geodetic distance inverse solution Andoyer-Lambert formulae generalization Forsyth method for geodesics(Corrected) Geodetic formulae(latitude, distance, azimuths) Geodesic approximations(spheroid)						

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2. Loran
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4. Sphere-Spheroidal Geometry

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